# SINGULAR FIOs IN SAR IMAGING, II: TRANSMITTER AND RECEIVER AT DIFFERENT SPEEDS

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Abstract. In this article, we consider two particular bistatic cases which arise in Synthetic Aperture Radar (SAR) imaging: when the transmitter and receiver are moving in the same direction or in the opposite direction and with different speeds. In both cases, we classify the forward operator $\mathcal{F}$ as an FIO with fold/blowdown singularities. Next we analyze the normal operator $\mathcal{F}^*\mathcal{F}$ in both cases (where $\mathcal{F}^*$ is the $L^2$ adjoint of $\mathcal{F}$). When the transmitter and receiver move in the same direction, we prove that $\mathcal{F}^*\mathcal{F}$ belongs to a class of distributions associated to two cleanly intersecting Lagrangians, $I^{p,l}(\Lambda_1, \Lambda_2)$. When they move in opposite directions, $\mathcal{F}^*\mathcal{F}$ is a sum of such operators. In both cases artifacts appear and we show that they are as strong as the bona-fide part of the image.

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## 1. Introduction

Synthetic Aperture Radar (SAR) makes possible high-resolution imaging in a variety of contexts. Applications include imaging the Earth's terrain, monitoring forestry bio-mass, aircraft identification, etc.

Although it is possible to collect enough data so that SAR images are reliable, it is often not practical to collect and process large amounts of data, especially when the image is needed in (almost) real time. Therefore, it may be the case that just a single sweep (instead of multiple sweeps) over the scene to be imaged is made and image is reconstructed based on this limited data.

It is reasonable to investigate the possibility of obtaining acceptable images using limited data and also, perhaps more importantly, study what artifacts can be expected when such SAR data is processed by conventional backprojection methods. Indeed, the results of our work [1] show that in the case of bistatic SAR, where the transmitter and receiver are moving with equal speed in opposite directions, there are unavoidable artifacts in the reconstructed image of the Earth's surface.

The current work continues on from [1] by analyzing the situation where the transmitter and receiver are either moving in the same direction or opposite direction and with different speeds. When the transmitter and receiver are moving in the same direction, we show that the scattering operator $\mathcal{F}$ that models the SAR data is a Fourier Integral Operator (FIO). The canonical relation $\mathcal{C}_\mathcal{F}$ of $\mathcal{F}$ is a Lagrangian submanifold of a product space $T^*X \times T^*Y$ and we show that the natural projections $\pi_L : \mathcal{C}_\mathcal{F} \to T^*Y, \pi_R : \mathcal{C}_\mathcal{F} \to T^*X$ drop rank along a smooth hypersurface $\Sigma_1$. As a consequence of this, the image that one obtains using standard backprojection has artifacts which are just as strong as the bona-fide part of the image. When the transmitter and receiver move in opposite directions, a similar situation occurs, except that the projections now drop rank over the union of a pair of smooth hypersurfaces and a similar statement regarding artifacts applies.

The images including artifacts are analyzed using the calculus of singular FIOs. There is extensive prior literature involving the use of microlocal techniques in the study of generalized Radon transforms, integral geometry, scattering theory and harmonic analysis [15, 13, 19, 14, 10, 12, 11, 8, 9, 22, 23, 24, 25, 3, 4, 5, 6, 18, 27, 7, 26]. We show that in the case of transmitter and receiver moving in the same direction, the normal operator $\mathcal{F}^*\mathcal{F}$ (an operator that plays a central role in the context of image analysis) has a distribution kernel belonging to the paired Lagrangian distribution class $I^{p,l}(\triangle, C)$ where $\triangle$ is the identity relation, $C$ is the graph of a simple reflection map and the orders $p, l$ of the operator are such that genuine scatterers and artifacts belong to the same Sobolev space. This result is valid even if the transmitter is stationary, for example, when the transmitter is a fixed radio tower and the receiver is a drone.

When the transmitter and receiver move in opposite directions, the analysis is considerably more complicated; see Theorem 2.6. As a special case of this result, we prove that choosing a suitable filtering of the data (Statements 1 and 3 of Theorem 2.6) shows that the canonical relation that causes the artifacts is the graph of an involution. On the other hand filtering the data as in Statement 2 of Theorem 2.6 shows that the backprojection adds two artifacts and the normal operator $\mathcal{G}^*\mathcal{G}$ is a sum of operators belonging to $I^{p,l}$ classes (where the forward operator for this case is denoted $\mathcal{G}$). In all these situations, we show that the additional artifacts are just as strong as the bona-fide part of the image. Finally, one can consider beam forming, in which certain portions of the ground are selectively illuminated; see Theorem 5.8. In this case, we show that the normal operator $\mathcal{G}^*\mathcal{G}$ still belongs to $I^{p,l}(\Delta, C)$ but $C$ is a two-sided fold canonical relation.

## 2. Statements of the main results

2.1. **The linearized scattering model.** For simplicity, we assume that both the transmitter and receiver are at the same height $h > 0$ above the ground at all times and that the transmitter and receiver move at constant but different speeds along a line parallel to the $x$ axis. Let

$$\gamma_T(s) = (\alpha s, 0, h) \qquad \gamma_R(s) = (s, 0, h) \tag{2.1}$$

for $s \in (0, \infty)$ be the trajectories of the transmitter and receiver respectively.

The case $\alpha = -1$ corresponds to the common midpoint problem which was fully analyzed in [1]. Therefore we will assume $\alpha \neq -1$. We also assume $\alpha \neq 1$ since this corresponds to the monostatic case which has also been fully analyzed in earlier works; [22, 3, 4].

The linearized model for the scattered signal we will use in this article is

$$\int e^{-\mathrm{i}\omega\left(t - \frac{1}{c_0}R(s,x)\right)} a_0(s, x, \omega) V(x) \mathrm{d}x \mathrm{d}\omega$$

for $(s, t) \in (0, \infty) \times (0, \infty)$, where $V(x) = V(x_1, x_2)$ is the function modeling the object on the ground, and

$$R(s, x) = \|\gamma_T(s) - x\| + \|x - \gamma_R(s)\|$$

is the bistatic distance–the sum of the distance from the transmitter to the scatterer and from the scatterer to the receiver, $c_0$ is the speed of electromagnetic wave in free-space and the amplitude term $a_0$ is given by

$$a_0(s, x, \omega) = \frac{\omega^2 p(\omega)}{16\pi^2 \|\gamma_T(s) - x\| \|\gamma_R(s) - x\|}.$$

This function includes terms that take into account the transmitted waveform and geometric spreading factors.

From now on, we denote the $(s, t)$ space by $Y = (0, \infty)^2$ and the $(x_1, x_2)$ space by $X = \mathbb{R}^2$.

For simplicity, we will assume that $c_0 = 1$. Because the ellipsoidal wavefronts do not meet the ground for

$$t < \sqrt{(\alpha - 1)^2 s^2 + 4h^2},$$

there is no signal for such $t$. As we will see, our method cannot image the point on the ground directly "between" the transmitter and receiver (see the proof of Theorem 2.1 in Section 4). Given transmitter and receiver positions $\alpha s$ and $s$ respectively, such a point on the ground has coordinates $\left(\frac{(\alpha+1)s}{2}, 0\right)$. Note that this point on the $x$-axis corresponds to $t = \sqrt{(\alpha-1)^2 s^2 + 4h^2}$. For these two reasons, we multiply $a_0$ by a cutoff function $f$ that is zero in a neighborhood of

$$\left\{(s,t) \,:\, s > 0,\ 0 < t \leq \sqrt{(\alpha-1)^2 s^2 + 4h^2}\right\}.$$

In addition, to be able to compose our forward operator and its adjoint, we further assume that $f$ is compactly supported and equal to 1 in a neighborhood of a suitably large compact subset of

$$\{(s,t) : s > 0, \sqrt{(\alpha-1)^2 s^2 + 4h^2} < t < \infty\}.$$

We let $f \cdot a_0 = a$, and this gives us the data

$$\mathcal{F}V(s,t) := \int e^{-i\omega(t - \|x-\gamma_T(s)\| - \|x-\gamma_R(s)\|)} a(s,t,x,\omega) V(x) \mathrm{d}x \mathrm{d}\omega. \tag{2.2}$$

We require additional cutoffs for our analysis to work for the case of $\alpha < 0$ (see Remarks 2.5 and 5.3).

Throughout the article we use the following notation

$$\begin{aligned} A = A(s,x) = \|x - \gamma_T(s)\| &= \sqrt{(x_1 - \alpha s)^2 + x_2^2 + h^2} \\ B = B(s,x) = \|x - \gamma_R(s)\| &= \sqrt{(x_1 - s)^2 + x_2^2 + h^2}. \end{aligned} \tag{2.3}$$

and we define the ellipse

$$E(s,t) = \left\{x \in \mathbb{R}^2 \,:\, A(s,x) + B(s,x) = t\right\} \tag{2.4}$$

We assume that the amplitude function $a \in S^2$, that is, it satisfies the following estimate: For every compact $K \subset Y \times X$ and for every non-negative integer $\delta$ and for every 2-index $\beta = (\beta_1, \beta_2)$ and $\lambda$, there is a constant $c$ such that

$$|\partial_\omega^\delta \partial_s^{\beta_1} \partial_t^{\beta_2} \partial_x^\lambda a(s,t,x,\omega)| \leq c(1 + |\omega|)^{2-\delta}. \tag{2.5}$$

This assumption is satisfied if the transmitted waveform from the antenna is approximately a Dirac delta distribution, and this is a standard assumption in the field.

2.2. **Transmitter and receiver moving in the same direction: $\alpha \geq 0$.** The case $\alpha \geq 0$ corresponds to the situation when the transmitter and receiver are traveling in the same direction or when the transmitter is stationary ($\alpha = 0$). For $\alpha \geq 0$, we refer to the forward operator by $\mathcal{F}$. We show that for the case $\alpha \geq 0$, the operator $\mathcal{F}$ in (2.2) is a Fourier integral operator (FIO) of order $\frac{3}{2}$ and study the properties of the natural projection maps from the canonical relation of $\mathcal{F}$. We have the following results.

**Theorem 2.1.** *Let $\mathcal{F}$ be the operator in* (2.2) *for $\alpha \geq 0$.*

*(1) $\mathcal{F}$ is an FIO of order* $3/2$.

*(2) The canonical relation $\mathcal{C}_\mathcal{F} \subset T^*Y \setminus \mathbf{0} \times T^*X \setminus \mathbf{0}$ associated to $\mathcal{F}$ is given by*

$$\begin{aligned} \mathcal{C}_\mathcal{F} = \Bigg\{ \Bigg( & s, t, -\omega \left( \frac{x_1 - \alpha s}{A} \alpha + \frac{x_1 - s}{B} \right), \omega; \\ & x_1, x_2, \omega \left( \frac{x_1 - \alpha s}{A} + \frac{x_1 - s}{B} \right), \omega \left( \frac{x_2}{A} + \frac{x_2}{B} \right) \Bigg) \\ & : \omega \neq 0,\ t = A + B \Bigg\}. \end{aligned} \tag{2.6}$$

*where $A = A(s, x)$ and $B = B(s, x)$ are defined in (2.3). Furthermore, $(s, x_1, x_2, \omega)$ is a global parameterization of $\mathcal{C}_\mathcal{F}$.*

*(3) Denote the left and right projections from $\mathcal{C}_\mathcal{F}$ to $T^*Y \setminus \mathbf{0}$ and $T^*X \setminus \mathbf{0}$ by $\pi_L$ and $\pi_R$ respectively. Then $\pi_L$ and $\pi_R$ drop rank simply by one on the set*

$$\Sigma_1 = \{(s, x_1, x_2, \omega) \in \mathcal{C}_\mathcal{F} \,:\, x_2 = 0\}. \tag{2.7}$$

*(4) $\pi_L$ has a fold singularity along $\Sigma_1$ and $\pi_R$ has a blowdown singularity along $\Sigma_1$.*

We next analyze the imaging operator $\mathcal{F}^*\mathcal{F}$.

**Theorem 2.2.** *Let $\mathcal{F}$ be as in Theorem 2.1. Then $\mathcal{F}^*\mathcal{F} \in I^{3,0}(\Delta, C_1)$, where*

$$C_1 = \{(x_1, x_2, \xi_1, \xi_2; x_1, -x_2, \xi_1, -\xi_2) \,:\, (x, \xi) \in T^*X \setminus \mathbf{0}\} \tag{2.8}$$

*which is the graph of $\chi_1(x, \xi) = (x_1, -x_2, \xi_1, -\xi_2)$.*

*Remark* 2.3. Since $\mathcal{F}^*\mathcal{F} \in I^{3,0}(\Delta, C_1)$, using the properties of the $I^{p,l}$ classes [16], we have that microlocally away from $C_1$, $\mathcal{F}^*\mathcal{F}$ is in $I^3(\Delta \setminus C_1)$ and microlocally away from $\Delta$, $\mathcal{F}^*\mathcal{F} \in I^3(C_1 \setminus \Delta)$. This means that $\mathcal{F}^*\mathcal{F}$ has the same order on both $\Delta$ and $C_1$ which implies that the artifacts caused by $C_1$ have the same order as the singularities in $V$ that cause them.

2.3. **Transmitter and receiver moving in opposite directions: $\alpha < 0$.** When $\alpha < 0$, the transmitter and receiver travel away from each other, and we refer to the forward operator by $\mathcal{G}$.

2.3.1. *Further preliminary modifications of the scattered data.* In the case when $\alpha < 0$, we further modify the operator $\mathcal{F}V$ considered in Section 2.1.

Our method cannot image a neighborhood of two points on the ground for a given transmitter and receiver positions in addition to the points muted by the cutoff function $f$ in Section 2.1. Therefore we modify or pre-process the receiver data further such that the contribution to it from a neighborhood of these two points is 0. The two points on the $x_1$-axis that we would like to avoid are of the form $\left(x_1^\pm, 0\right)$, where

$$x_1^+ = \frac{2\alpha s}{\alpha + 1} + \sqrt{-\alpha \frac{(\alpha - 1)^2}{(\alpha + 1)^2} s^2 - h^2}, \tag{2.9}$$

$$x_1^- = \frac{2\alpha s}{\alpha + 1} - \sqrt{-\alpha \frac{(\alpha - 1)^2}{(\alpha + 1)^2} s^2 - h^2} \tag{2.10}$$

as explained in Remark 2.5. We define a smooth mute function $g(s, t)$ that is identically 0 if the ellipse $E(s, t)$ is near one of the points $(x_1^\pm, 0)$; for each $s$, the corresponding values of $t$ are

$$t_s^\pm = A\left(s, (x_1^\pm, 0)\right) + B\left(s, (x_1^\pm, 0)\right) \tag{2.11}$$

where $A$ and $B$ are given by (2.3). The points $t_s^\pm$ are given explicitly in Appendix B.

With the function $g$, we modify $\mathcal{F}$ in (2.2) by replacing $a$ by $g \cdot a$ and call it $a$ again. Throughout this section, corresponding to the case $\alpha < 0$, we will designate the operator as $\mathcal{G}$. That is, we have

$$\mathcal{G}V(s,t) := \int e^{-i\omega(t - \|x-\gamma_T(s)\| - \|x-\gamma_R(s)\|)} a(s,t,x,\omega) V(x) \mathrm{d}x \mathrm{d}\omega, \tag{2.12}$$

where $a$ takes into account the cutoff functions $f$ in Section 2.1 and the function $g$ defined in the last paragraph.

**Theorem 2.4.** *Let $\mathcal{G}$ be the operator given in* (2.12) *for $\alpha < 0$. Then*

*(1) $\mathcal{G}$ is an FIO of order $\frac{3}{2}$*

*(2) The canonical relation $\mathcal{C}_\mathcal{G}$ associated to $\mathcal{G}$ is given by* (2.6) *with global parameterization $(s, x_1, x_2, \omega)$.*

*(3) The left and right projections $\pi_L$ and $\pi_R$ respectively from $\mathcal{C}_\mathcal{G}$ drop rank simply by one on the set $\Sigma = \Sigma_1 \cup \Sigma_2$ where $\Sigma_1$ is given by* (2.7) *and*

$$\Sigma_2 = \left\{ (s,x,\omega) \in \mathcal{C}_\mathcal{G} \,:\, \frac{\alpha}{A^2} + \frac{1}{B^2} = 0,\ x_2 \neq 0 \right\} \tag{2.13}$$

$$= \left\{ (s,x,\omega) \in \mathcal{C}_\mathcal{G} \,:\, \left( x_1 - \frac{2\alpha s}{\alpha+1} \right)^2 + x_2^2 = -\alpha s^2 \frac{(\alpha-1)^2}{(\alpha+1)^2} - h^2,\ x_2 \neq 0 \right\} \tag{2.14}$$

*(4) $\pi_L$ has a fold singularity along $\Sigma$.*

*(5) $\pi_R$ has a blowdown singularity along $\Sigma_1$ and a fold singularity along $\Sigma_2$.*

For convenience, we denote, for each $s$, the projection of the part of $\Sigma_2$ above $s$ to $\mathbb{R}^2$ (the projection to the base of $\pi_R\left(\Sigma_2\big|_s\right)$) by $\Sigma_{2,X}(s)$, and this is the circle described in (2.14) and in Appendix B. It can be written

$$\Sigma_{2,X}(s) = \left\{ x \,:\, \frac{\alpha}{A^2(s,x)} + \frac{1}{B^2(s,x)} = 0,\ x_2 \neq 0 \right\}. \tag{2.15}$$

*Remark* 2.5. From Equation (2.14) we have that $\Sigma_{2,X}(s)$ is a circle of radius $\sqrt{-\alpha s^2 \frac{(\alpha-1)^2}{(\alpha+1)^2} - h^2}$ and centered at $(2\alpha s/(\alpha+1), 0)$.

Now we can explain why we need to cutoff the data for ellipses near the two points given by (2.9)-(2.10). Since $\pi_R(\Sigma_1)$ intersects $\pi_R(\Sigma_2)$ above these two points, $\pi_R$ drops rank by two above these points. So, we mute data near $(s, t_s^{\pm})$ given by (2.11). We will precisely describe this mute, $g$ in Remark 5.3.

We now analyze the imaging operator $\mathcal{G}^*\mathcal{G}$. Unlike the case $\alpha \geq 0$, this case is more complicated and we consider several restricted transforms.

**Theorem 2.6.** *Let $-1 \neq \alpha \leq 0$ and $\mathcal{G}$ be the operator in* (2.12) *and let*

$$s_0 = \frac{h(\alpha+1)}{\sqrt{-\alpha}(\alpha-1)} \tag{2.16}$$

*Then the following hold:*

*(1) Let $O_1 = \{(s,t) : 0 < s < s_0 \text{ and } 0 < t < \infty\}$ and let $r_1$ be a smooth cutoff function that is compactly supported in $O_1$. Consider the operator $\mathcal{G}$ defined in* (2.12) *with the amplitude function $a$ replaced by $r_1 \cdot a$. Then $\mathcal{G}^*\mathcal{G} \in I^{3,0}(\Delta, C_1)$ where $C_1$ is defined in* (2.8).

*(2) Let $O_2 = \{(s,t) : s_0 < s < \infty \text{ and } t_s^- < t < t_s^+\}$ where $t_s^{\pm}$ is defined in* (2.11). *Let $r_2$ be a smooth cutoff function and compactly supported in $O_2$. Consider the operator $\mathcal{G}$ defined in* (2.12) *with the amplitude function $a$ replaced by $r_2 \cdot a$. Then $\mathcal{G}^*\mathcal{G} \in I^{3,0}(\Delta, C_1) + I^{3,0}(\Delta, C_2) + I^{3,0}(C_1, C_2)$ where $C_2$ is a two-sided fold given by* (5.1).

*(3) Let $O_3 = \{(s,t) : s_0 < s < \infty$ and $t < t_s^-$ or $t > t_s^+\}$ with $t_s^\pm$ defined in* (2.11). *Let $r_3$ be a smooth cutoff function compactly supported in $O_3$. Consider the operator $\mathcal{G}$ defined in* (2.12) *with the amplitude function $a$ replaced by $r_3 \cdot a$. Then $\mathcal{G}^*\mathcal{G} \in I^{3,0}(\Delta, C_1)$.*

*Remark* 2.7. Using the properties of the $I^{p,l}$ classes for case (2) of the theorem,

$$\mathcal{G}^*\mathcal{G} \in I^{3,0}(\Delta, C_1) + I^{3,0}(\Delta, C_2) + I^{3,0}(C_1, C_2)$$

implies that singularities in the reconstruction will show up because of $C_1$ (reflection in the $x_1$ axis) and because of $C_2$ (a 2-sided fold). Furthermore, the added artifacts will have the same order as the singularities in $V$ that cause them.

## 3. Preliminaries: Singularities and $I^{p,l}$ classes

Here we introduce the classes of distributions and singular FIO we will use to analyze the forward operators $\mathcal{F}$ and $\mathcal{G}$ and the normal operators $\mathcal{F}^*\mathcal{F}$ and $\mathcal{G}^*\mathcal{G}$.

**Definition 3.1.** [11] Let $M$ and $N$ be manifolds of dimension $n$ and let $f : M \to N$ be $C^\infty$. Define $\Sigma = \{m \in M : \det(df)_m = 0\}$.

1. $f$ drops rank by one simply on $\Sigma$ if for each $m_0 \in \Sigma$,
   rank $(df)_{m_0} = n-1$ and $d(\det(df)_{m_0}) \neq 0$.
2. $f$ has a *Whitney fold* along $\Sigma$ if $f$ is a local diffeomorphism away from $\Sigma$ and $f$ drops rank by one simply on $\Sigma$, so that $\Sigma$ is a smooth hypersurface and ker $(df)_{m_0} \not\subset T_{m_0}\Sigma$ for every $m_0 \in \Sigma$.
3. $f$ is a *blow-down* along $\Sigma$ if $f$ is a local diffeomorphism away from $\Sigma$ and $f$ drops rank by one simply on $\Sigma$, so that $\Sigma$ is a smooth hypersurface and $\ker(df)_{m_0} \subset T_{m_0}(\Sigma)$ for every $m_0 \in \Sigma$.

**Definition 3.2** ([19])**.** A smooth canonical relation $C$ for which both projections $\pi_L$ and $\pi_R$ have only (Whitney) fold singularities, is called a *two-sided fold* or a *folding canonical relation.*

This notion was first introduced by Melrose and Taylor [19], who showed the existence of a normal form in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$.

**Theorem 3.3** ([19])**.** *If dim $X = n$ dim $Y = n$ and $C \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$ is a two-sided fold, then microlocally there are homogeneous canonical transformations, $\chi_1 : T^*X \to T^*\mathbb{R}^n$ and $\chi_2 : T^*Y \to T^*\mathbb{R}^n$, such that $(\chi_1 \times \chi_2)(C) \subseteq C_0$, near $\xi_2 \neq 0$ where, $C_0 = N^*\{x_2 - y_2 = (x_1 - y_1)^3;\ x_i = y_i,\ 3 \le i \le n\}$.*

We now define $I^{p,l}$ classes. They were first introduced by Melrose and Uhlmann [20], Guillemin and Uhlmann [16] and Greenleaf and Uhlmann [12] and they have been used in the study of SAR imaging [22, 3, 4, 18, 1].

**Definition 3.4.** Two submanifolds $M$ and $N$ intersect *cleanly* if $M \cap N$ is a smooth submanifold and $T(M \cap N) = TM \cap TN$.

Consider two spaces $X$ and $Y$ and let $\Lambda_0$ and $\Lambda_1$ and $\tilde\Lambda_0$ and $\tilde\Lambda_1$ be Lagrangian submanifolds of the product space $T^*X \times T^*Y$. If they intersect cleanly, $(\tilde\Lambda_0, \tilde\Lambda_1)$ and $(\Lambda_0, \Lambda_1)$ are equivalent in the sense that there is, microlocally, a canonical transformation $\chi$ which maps $(\Lambda_0, \Lambda_1)$ into $(\tilde\Lambda_0, \tilde\Lambda_1)$ and $\chi(\Lambda_0 \cap \Lambda_1) = (\tilde\Lambda_0 \cap \tilde\Lambda_1)$. This leads us to the following model case.

*Example.* Let $\tilde\Lambda_0 = \Delta_{T^*\mathbb{R}^n} = \{(x, \xi; x, \xi) : x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n \setminus \mathbf{0}\}$ be the diagonal in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ and let $\tilde\Lambda_1 = \{(x', x_n, \xi', 0; x', y_n, \xi', 0) : x' \in \mathbb{R}^{n-1},\ \xi' \in \mathbb{R}^{n-1} \setminus \mathbf{0}\}$. Then, $\tilde\Lambda_0$ intersects $\tilde\Lambda_1$ cleanly in codimension 1.

Now we define the class of product-type symbols $S^{p,l}(m, n, k)$.

**Definition 3.5.** $S^{p,l}(m,n,k)$ is the set of all functions $a(z;\xi,\sigma)\in C^\infty(\mathbb{R}^m\times\mathbb{R}^n\setminus\mathbf{0}\times\mathbb{R}^k)$ such that for every $K\subset\mathbb{R}^m$ and every $\alpha\in\mathbb{Z}_+^m,\beta\in\mathbb{Z}_+^n,\delta\in\mathbb{Z}_+^k$ there is $c_{K,\alpha,\beta}$ such that

$$|\partial_z^\alpha\partial_\xi^\beta\partial_\sigma^\delta a(z,\xi,\sigma)|\leq c_{K,\alpha,\beta}(1+|\xi|)^{p-|\beta|}(1+|\sigma|)^{l-|\delta|}$$

for all $(z,\xi,\tau)\in K\times\mathbb{R}^n\setminus\mathbf{0}\times\mathbb{R}^k$.

Since any two sets of cleanly intersecting Lagrangians are equivalent, we first define $I^{p,l}$ classes for the case in Example 3.

**Definition 3.6.** [16] Let $I^{p,l}(\tilde\Lambda_0,\tilde\Lambda_1)$ be the set of all distributions $u$ such that $u=u_1+u_2$ with $u_1\in C_0^\infty$ and

$$u_2(x,y)=\int e^{i((x'-y')\cdot\xi'+(x_n-y_n-s)\cdot\xi_n+s\cdot\sigma)}a(x,y,s;\xi,\sigma)d\xi d\sigma ds$$

with $a\in S^{p',l'}$ where $p'=p-\frac{n}{2}+\frac{1}{2}$ and $l'=l-\frac{1}{2}$.

This allows us to define the $I^{p,l}(\Lambda_0,\Lambda_1)$ class for any two cleanly intersecting Lagrangians in codimension 1 using the microlocal equivalence with the case in Example 3.

**Definition 3.7.** [16] Let $I^{p,l}(\Lambda_0,\Lambda_1)$ be the set of all distributions $u$ such that $u=u_1+u_2+\sum v_i$ where $u_1\in I^{p+l}(\Lambda_0\setminus\Lambda_1)$, $u_2\in I^p(\Lambda_1\setminus\Lambda_0)$, the sum $\sum v_i$ is locally finite and $v_i=Aw_i$ where $A$ is a zero order FIO associated to $\chi^{-1}$, the canonical transformation from above, and $w_i\in I^{p,l}(\tilde\Lambda_0,\tilde\Lambda_1)$.

This class of distributions is invariant under FIOs associated to canonical transformations which map the pair $(\Lambda_0,\Lambda_1)$ to itself, whilst also preserving the intersection. By definition, $F\in I^{p,l}(\Lambda_0,\Lambda_1)$ if its Schwartz kernel belongs to $I^{p,l}(\Lambda_0,\Lambda_1)$. If $F\in I^{p,l}(\Lambda_0,\Lambda_1)$ then $F\in I^{p+l}(\Lambda_0\setminus\Lambda_1)$ and $F\in I^p(\Lambda_1\setminus\Lambda_0)$ [16]. Here by $F\in I^{p+l}(\Lambda_0\setminus\Lambda_1)$, we mean that the Schwartz kernel of $F$ belongs to $I^{p+l}(\Lambda_0)$ microlocally away from $\Lambda_1$.

To show that a distribution belongs to $I^{p,l}$ class we use the iterated regularity property:

**Theorem 3.8** ([12, Proposition 1.35])**.** *If $u\in\mathcal{D}'(X\times Y)$ then $u\in I^{p,l}(\Lambda_0,\Lambda_1)$ if there is an $s_0\in\mathbb{R}$ such that for all first order pseudodifferential operators $P_i$ with principal symbols vanishing on $\Lambda_0\cup\Lambda_1$, we have $P_1P_2\dots P_ru\in H^{s_0}_{loc}$.*

In section 5, we will use the following theorem.

**Theorem 3.9** ([3, 21])**.** *If $F$ is a FIO of order $m$ whose canonical relation is a two-sided fold then $F^*F\in I^{2m,0}(\Delta,\tilde C)$ where $\tilde C$ is another two-sided fold.*

## 4. Analysis of the operator $\mathcal{F}$ and the imaging operator $\mathcal{F}^*\mathcal{F}$ for $\alpha\geq 0$

In this section, we prove Theorems 2.1 and 2.2.

*Proof of Theorem 2.1.* We first prove that

$$\varphi:=-\omega\left(t-\sqrt{(x_1-\alpha s)^2+x_2^2+h^2}-\sqrt{(x_1-s)^2+x_2^2+h^2}\right)$$

is a non-degenerate phase function. We have that $\varphi$ is a phase function because $\nabla_x\varphi\neq 0$ at points where the amplitude of the operator $\mathcal{F}$ is elliptic. The differential $\nabla_x\varphi$ vanishes at a point on the ground directly "between" the source and receiver and this point is given by $\left(\frac{(\alpha+1)s}{2},0\right)$. However, in a neighborhood of such points the amplitude $a$ vanishes due to the cutoff function $f$ in the definition of $\mathcal{F}$ given in (2.2). Also we have that $\nabla_{s,t}\varphi$ is nowhere 0 since $\omega\neq 0$. The same reason that $\nabla_x\varphi$ is non-vanishing at points where the amplitude $a$ is elliptic also gives that $\varphi$ is non-degenerate. Since $a$ satisfies an amplitude estimate, $\mathcal{F}$ is an FIO [2]. Finally since $a$ is of order 2, the order of the FIO is 3/2 [2, Definition 3.2.2]. By definition [17, Equation (3.1.2)]

$$\mathcal{C}_{\mathcal{F}} = \Big\{(s,t,\partial_{s,t}\varphi(x,s,t,\omega));\ (x,-\partial_x\varphi(x,s,t));\ \partial_\omega\varphi(x,s,t,\omega)=0\Big\}.$$

This establishes (2.6). Furthermore, it is easy to see that $(x_1,x_2,s,\omega)$ is a global parametrization of $\mathcal{C}_{\mathcal{F}}$.

Now we prove the claims about the canonical left and right projections from $\mathcal{C}_{\mathcal{F}}$, the final parts of Theorem 2.1. In the parameterization of $\mathcal{C}_{\mathcal{F}}$, we have

$$\pi_L(x_1,x_2,s,\omega) = \left(s, A+B, -\left(\frac{x_1-\alpha s}{A}\alpha + \frac{x_1-s}{B}\right)\omega, -\omega\right)$$

and the derivative is

$$(d\pi_L) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{x_1-\alpha s}{A} + \frac{x_1-s}{B} & \frac{x_2}{A}+\frac{x_2}{B} & * & 0 \\ -\omega\left(\frac{x_2^2+h^2}{A^3}\alpha + \frac{x_2^2+h^2}{B^3}\right) & \omega\left(\frac{\alpha(x_1-\alpha s)x_2}{A^3} + \frac{(x_1-s)x_2}{B^3}\right) & * & * \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then

$$\det(d\pi_L) = -\omega x_2\left(\frac{\alpha}{A^2}+\frac{1}{B^2}\right)\left(1+\frac{(\gamma_T-x)\cdot(\gamma_R-x)}{AB}\right). \tag{4.1}$$

The third term would be zero when the unit vectors $(\gamma_T(s)-x)/A$ and $(\gamma_R(s)-x)/B$ point in opposite directions, but this cannot happen since the transmitter and receiver are above the plane of the Earth. Also since $\alpha>0$, $\left(\frac{\alpha}{A^2}+\frac{1}{B^2}\right)\neq 0$. Hence, this determinant vanishes to first order when $x_2=0$. This corresponds to $\Sigma_1$ given in (2.7).

On $\Sigma_1$ the kernel of $(d\pi_L)$ is spanned by $\frac{\partial}{\partial x_2}\not\subset T\Sigma_1$. So $\pi_L$ has a fold singularity along $\Sigma_1$.

Similarly, we have,

$$\pi_R(x_1,x_2,s,\omega) = \left(x_1,x_2,-\left(\frac{x_1-\alpha s}{A}+\frac{x_1-s}{B}\right)\omega, -\left(\frac{x_2}{A}+\frac{x_2}{B}\right)\omega\right).$$

Then

$$(d\pi_R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & \omega(\frac{x_2^2+h^2}{A^3}\alpha + \frac{x_2^2+h^2}{B^3}) & -(\frac{x_1-\alpha s}{A}+\frac{x_1-s}{B}) \\ * & * & -\omega(\frac{(x_1-\alpha s)x_2}{A^3}\alpha + \frac{(x_1-s)x_2}{B^3}) & -(\frac{x_2}{A}+\frac{x_2}{B}) \end{pmatrix}$$

has the same determinant as $(d\pi_L)$.Therefore $\pi_R$ drops rank simply by one on $\Sigma_1$. On $\Sigma_1$, the kernel of $\pi_R$ is spanned by $\frac{\partial}{\partial\omega}$ and $\frac{\partial}{\partial s}$ which are tangent to $\Sigma_1$. Thus $\pi_R$ has a blowdown singularity along $\Sigma_1$.

□

Next we analyze the imaging operator $\mathcal{F}^*\mathcal{F}$. We have the following integral representation for $\mathcal{F}^*\mathcal{F}$:

$$\mathcal{F}^*\mathcal{F}V(x) = \int e^{\mathrm{i}\tilde{\phi}(x,s,t,\omega,\tilde{\omega},y)}\overline{a(s,t,x,\omega)}a(s,t,y,\tilde{\omega})V(y)\mathrm{d}s\mathrm{d}t\mathrm{d}\omega\mathrm{d}\tilde{\omega}\mathrm{d}y,$$

where

$$\begin{aligned}\tilde{\phi} =&\big(\omega\left(t-(\|x-\gamma_T(s)\|+\|x-\gamma_R(s)\|)\right) \\ &-\tilde{\omega}\left(t-(\|y-\gamma_T(s)\|+\|y-\gamma_R(s)\|)\right)\big)\end{aligned}$$

After an application of the method of stationary phase in $t$ and $\tilde{\omega}$, the Schwartz kernel of this operator becomes

$$K(x,y) = \int e^{i\Phi(y,s,x,\omega)}\tilde{a}(y,s,x,\omega)\,\mathrm{d}s\mathrm{d}\omega. \tag{4.2}$$

where

$$\begin{aligned}\Phi(y,s,x,\omega) =&\omega\big(\,\|y-\gamma_T(s)\| + \|y-\gamma_R(s)\| \\ &\qquad - (\|x-\gamma_T(s)\| + \|x-\gamma_R(s)\|)\big).\end{aligned} \tag{4.3}$$

Note that $\tilde{a} \in S^4$ since we have assumed that $a \in S^2$.

**Proposition 4.1.** *Let $\alpha \geq 0$. The wavefront set of the kernel $K$ of $\mathcal{F}^*\mathcal{F}$ satisfies*

$$WF'(K) \subset \Delta \cup C_1,$$

*where $\Delta$ is the diagonal in $T^*X \times T^*X$, and $C_1$ is given by (2.8). We have that $\Delta$ and $C_1$ intersect cleanly in codimension 2 in $\Delta$ or $C_1$.*

*Proof.* Let $(s,t,\sigma,\tau;y,\eta) \in \mathcal{C}_{\mathcal{F}}$. Then we have

$$\begin{aligned}
t &= \sqrt{(y_1-\alpha s)^2 + y_2^2 + h^2} + \sqrt{(y_1-s)^2 + y_2^2 + h^2}\\
\sigma &= \tau\left(\frac{y_1-\alpha s}{\sqrt{(y_1-\alpha s)^2 + y_2^2 + h^2}}\alpha + \frac{y_1-s}{\sqrt{(y_1-s)^2 + y_2^2 + h^2}}\right)\\
\eta_1 &= \tau\left(\frac{y_1-\alpha s}{\sqrt{(y_1-\alpha s)^2 + y_2^2 + h^2}} + \frac{y_1-s}{\sqrt{(y_1-s)^2 + y_2^2 + h^2}}\right)\\
\eta_2 &= \tau\left(\frac{y_2}{\sqrt{(y_1-\alpha s)^2 + y_2^2 + h^2}} + \frac{y_2}{\sqrt{(y_1-s)^2 + y_2^2 + h^2}}\right)
\end{aligned} \tag{4.4}$$

and $(x,\xi;s,t,\sigma,\tau) \in C^t$ implies

$$\begin{aligned}
t &= \sqrt{(x_1-\alpha s)^2 + x_2^2 + h^2} + \sqrt{(x_1-s)^2 + x_2^2 + h^2}\\
\sigma &= \tau\left(\frac{x_1-\alpha s}{\sqrt{(x_1-\alpha s)^2 + x_2^2 + h^2}}\alpha + \frac{x_1-s}{\sqrt{(x_1-s)^2 + x_2^2 + h^2}}\right)\\
\xi_1 &= \tau\left(\frac{x_1-\alpha s}{\sqrt{(x_1-\alpha s)^2 + x_2^2 + h^2}} + \frac{x_1-s}{\sqrt{(x_1-s)^2 + x_2^2 + h^2}}\right)\\
\xi_2 &= \tau\left(\frac{x_2}{\sqrt{(x_1-\alpha s)^2 + x_2^2 + h^2}} + \frac{x_2}{\sqrt{(x_1-s)^2 + x_2^2 + h^2}}\right)
\end{aligned} \tag{4.5}$$

From the first two relations in (4.4) and (4.5), we have

$$\begin{aligned}
&\sqrt{(y_1-\alpha s)^2 + y_2^2 + h^2} + \sqrt{(y_1-s)^2 + y_2^2 + h^2}\\
&\quad = \sqrt{(x_1-\alpha s)^2 + x_2^2 + h^2} + \sqrt{(x_1-s)^2 + x_2^2 + h^2}
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
&\frac{y_1-\alpha s}{\sqrt{(y_1-\alpha s)^2 + y_2^2 + h^2}}\alpha + \frac{y_1-s}{\sqrt{(y_1-s)^2 + y_2^2 + h^2}}\\
&\quad = \frac{x_1-\alpha s}{\sqrt{(x_1-\alpha s)^2 + x_2^2 + h^2}}\alpha + \frac{x_1-s}{\sqrt{(x_1-s)^2 + x_2^2 + h^2}}.
\end{aligned} \tag{4.7}$$

We will use prolate spheroidal coordinates with foci $\gamma_R(s)$ and $\gamma_T(s)$ to solve for $x$ and $y$. We let

$$\begin{array}{ll} x_1 = \frac{1+\alpha}{2}s + \frac{1-\alpha}{2}s\cosh\rho\cos\phi & y_1 = \frac{1+\alpha}{2}s + \frac{1-\alpha}{2}s\cosh\rho'\cos\phi' \\ x_2 = \frac{1-\alpha}{2}s\sinh\rho\sin\phi\cos\theta & y_2 = \frac{1-\alpha}{2}s\sinh\rho'\sin\phi'\cos\theta' \\ x_3 = h + \frac{1-\alpha}{2}s\sinh\rho\sin\phi\sin\theta & y_3 = h + \frac{1-\alpha}{2}s\sinh\rho'\sin\phi'\sin\theta' \end{array} \tag{4.8}$$

with $\rho$ and $\rho'$ positive, $\phi$ and $\phi'$ in the interval $[0,\pi]$ and $\theta$ and $\theta'$ in $[0,2\pi]$. In this case $x_3 = 0$ and we use it to solve for $h$. Hence

$$A^2 = (x_1 - \alpha s)^2 + x_2^2 + h^2 = \frac{(1-\alpha)^2}{4}s^2(\cosh\rho + \cos\phi)^2$$

and

$$B^2 = (x_1 - s)^2 + x_2^2 + h^2 = \frac{(1-\alpha)^2}{4}s^2(\cosh\rho - \cos\phi)^2.$$

Noting that $s > 0$ and $\cosh\rho \pm \cos\phi > 0$, the first relation given by (4.6) in these coordinates becomes

$$s(\cosh\rho - \cos\phi) + s(\cosh\rho + \cos\phi) = s(\cosh\rho' - \cos\phi') + s(\cosh\rho' + \cos\phi')$$

from which we get

$$\cosh\rho = \cosh\rho' \Rightarrow \rho = \rho'.$$

The second relation, given by (4.7), becomes

$$\frac{\cosh\rho\cos\phi - 1}{\cosh\rho - \cos\phi} + \alpha\frac{\cosh\rho\cos\phi + 1}{\cosh\rho + \cos\phi} = \frac{\cosh\rho\cos\phi' - 1}{\cosh\rho - \cos\phi'} + \alpha\frac{\cosh\rho\cos\phi' + 1}{\cosh\rho + \cos\phi'}$$

After simplification we get

$$\begin{aligned}(\cos\phi - \cos\phi')[(\alpha+1)(\cosh^2\rho + \cos\phi\cos\phi') \\ - (\alpha-1)\cosh\rho(\cos\phi + \cos\phi')] = 0\end{aligned} \tag{4.9}$$

which implies either that

$$\cos\phi = \cos\phi' \text{ which implies } \phi = \phi' \tag{4.10}$$

(since we can assume $\phi \in [\pi, 2\pi]$ for points on the ground) or that

$$(\alpha+1)(\cosh^2\rho + \cos\phi\cos\phi') - (\alpha-1)\cosh\rho(\cos\phi + \cos\phi') = 0 = \frac{\alpha}{A\tilde{A}} + \frac{1}{B\tilde{B}} \tag{4.11}$$

where $\tilde{A}$ and $\tilde{B}$ are defined as in (2.3) but evaluated at $(s, y)$ and the third term in the equality is equivalent to the first term.

We consider Conditions (4.10) and (4.11) separately. First, assume Condition (4.10) holds. Then we have $\phi = \phi'$. In this case,

$$\cos\theta = \pm\sqrt{1 - \frac{h^2}{s^2\sinh^2\rho\sin^2\phi}} = \pm\cos\theta'$$

and note that $x_3 = 0$ implies that $\sin\phi \neq 0$. We also remark that it is enough to consider $\cos\theta = \cos\theta'$ as no additional relations are introduced by considering $\cos\theta = -\cos\theta'$ over the relations we now address. Now we go back to $x$ and $y$ coordinates. If $\theta = \theta'$ then $x_1 = y_1$, $x_2 = y_2, \xi_i = \eta_i$ for $i = 1, 2$. In this case, the composition, $\mathcal{C}_{\mathcal{F}}^t \circ \mathcal{C}_{\mathcal{F}} \subset \Delta = \{(x,\xi; x,\xi)\}$. If $\theta' = \pi - \theta$ then $x_1 = y_1$, $-x_2 = y_2, \xi_1 = \eta_1, -\xi_2 = \eta_2$. For these points the composition, $\mathcal{C}_{\mathcal{F}}^t \circ \mathcal{C}_{\mathcal{F}}$ is a subset of $C_1$ in (2.8). Note that (4.11) has no solutions for $\alpha \geq 0$. The statements about clean intersection are the same as the one given in [1]. This concludes the proof of Proposition 4.1.

□

*Proof of Theorem 2.2.* We will use the iterated regularity method (Theorem 3.8) to show that the kernel of $\mathcal{F}^*\mathcal{F} \in I^{3,0}(\Delta, C_1)$. We consider the generators of the ideal of functions that vanish on $\Delta \cup C_1$ [3]. These are given by

$$\begin{aligned} &\tilde{p}_1 = x_1 - y_1, \quad \tilde{p}_2 = x_2^2 - y_2^2, \quad \tilde{p}_3 = \xi_1 - \eta_1, \\ &\tilde{p}_4 = (x_2 - y_2)(\xi_2 + \eta_2), \quad \tilde{p}_5 = (x_2 + y_2)(\xi_2 - \eta_2), \\ &\tilde{p}_6 = \xi_2^2 - \eta_2^2. \end{aligned} \tag{4.12}$$

We show in Appendix A that each $\tilde{p}_i$ can be expressed as sums of products of $\partial_\omega\Phi$ and $\partial_s\Phi$ with smooth functions. Let $p_i = q_i\tilde{p}_i$, for $1 \leq i \leq 6$, where $q_1, q_2$ are homogeneous of degree 1 in $(\xi, \eta)$, $q_3, q_4$ and $q_5$ are homogeneous of degree 0 in $(\xi, \eta)$ and $q_6$ is homogeneous of degree $-1$ in $(\xi, \eta)$. Let $P_i$ be pseudodifferential operators with principal symbols $p_i$ for $1 \leq i \leq 6$. The $\tilde{p}_i$ and arguments using iterated regularity are similar to those used in [3, Thm. 1.6] and in [18, Thm. 4.3].

We use the same arguments as in [1] to show that the orders $p, l$ from $I^{p,l}(\Delta, C_1)$ are $p = 3$ and $l = 0$. □

## 5. Analysis of the forward operator $\mathcal{G}$ and the imaging operator $\mathcal{G}^*\mathcal{G}$ for $\alpha < 0$

In this section, we analyze the operator $\mathcal{G}$ in (2.12). In [1] we analyzed the case $\alpha = -1$. For the case with $\alpha < 0$, we make another simplification:

$$\text{Assume} \quad \alpha < -1.$$

If $\alpha \in (-1, 0)$, then we can reduce it to the case $\alpha < -1$ by using the diffeomorphisms $(x_1, x_2) \mapsto (-x_1, x_2)$ and $(s, t) \mapsto (s/|\alpha|, t)$.

We first prove Theorem 2.4.

*Proof of Theorem 2.4.* In the proof of this theorem, most of the statements are already proved in Theorem 2.1. We just prove the statements regarding the properties of the projection maps $\pi_L$ and $\pi_R$. Recall from the proof of Theorem 2.1 that

$$\mathrm{d}\pi_L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{x_1-\alpha s}{A} + \frac{x_1-s}{B} & \frac{x_2}{A} + \frac{x_2}{B} & * & 0 \\ -\omega\left(\frac{x_2^2+h^2}{A^3}\alpha + \frac{x_2^2+h^2}{B^3}\right) & \omega\left(\frac{\alpha(x_1-\alpha s)x_2}{A^3} + \frac{(x_1-s)x_2}{B^3}\right) & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\det \mathrm{d}\pi_L = \omega x_2\left(\frac{\alpha}{A^2} + \frac{1}{B^2}\right)\left(1 + \frac{(\gamma_T - x)\cdot(\gamma_R - x)}{AB}\right)$$

Clearly this determinant drops rank when the first term, $x_2 = 0$. This corresponds to $\Sigma_1$ given by (2.7).

The determinant also drops rank when the second term is zero, which can occur when $\alpha < 0$; this corresponds to $\Sigma_2$ given by (2.13). Note that $\pi_L$ drops rank by 2 at the intersection points of $\Sigma_1$ and $\Sigma_2$ (where $x_2 = 0$) but we exclude them using the cutoff function $g$ described preceding (2.11).

On $\Sigma_2$, using the first, second, and fourth row of $d\pi_L$, the kernel is $\frac{\partial}{\partial x_1} - \frac{\frac{x_1-\alpha s}{A}+\frac{x_1-s}{B}}{x_2(\frac{1}{A}+\frac{1}{B})}\frac{\partial}{\partial x_2}$ which applied to $\Sigma_2$ gives us $\frac{2s(\alpha-1)}{(\alpha+1)(\frac{1}{A}+\frac{1}{B})}(\frac{\alpha}{A} - \frac{1}{B})$. We have that $s \neq 0$ and $\alpha + 1 \neq 0$. If $\frac{\alpha}{A} - \frac{1}{B} = 0$ then using $\frac{\alpha}{A^2} + \frac{1}{B^2} = 0$ we get $\frac{\alpha(\alpha+1)}{A^2} = 0$ which is a contradiction. Thus $\ker(\mathrm{d}\pi_L) \not\subset T\Sigma_2$ which implies that $\pi_L$ has a fold singularity along $\Sigma_2$.

Similarly,

$$
\mathrm{d}\pi_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & -\omega(\frac{x_2^2+h^2}{A^3}\alpha + \frac{x_2^2+h^2}{B^3}) & (\frac{x_1-\alpha s}{A} + \frac{x_1-s}{B}) \\ * & * & \omega(\frac{(x_1-\alpha s)x_2}{A^3}\alpha + \frac{(x_1-s)x_2}{B^3}) & (\frac{x_2}{A} + \frac{x_2}{B}) \end{pmatrix}
$$

has the same determinant up to sign and so $\pi_R$ drops rank by one on $\Sigma$. On $\Sigma_2$, using the last row of $d\pi_R$, the kernel is $\frac{\partial}{\partial s} - \omega \frac{\frac{x_1-\alpha s}{A^3}\alpha + \frac{x_1-s}{B^3}}{\frac{1}{A}+\frac{1}{B}} \frac{\partial}{\partial \omega}$ which applied to $\Sigma_2$ gives $2\alpha(s - \frac{2x_1}{\alpha+1})$. If $s = \frac{2x_1}{\alpha+1}$ then from $\frac{\alpha}{A^2} + \frac{1}{B^2} = 0$ we obtain $s^2(\frac{\alpha-1}{2})^2 + x_2^2 + h^2 = 0$ which is a contradiction. Hence $\ker(\mathrm{d}\pi_R) \not\subset T\Sigma_2$ which implies that $\pi_R$ has a fold singularity along $\Sigma_2$ as well. This completes the proof of Theorem 2.4. ☐

**Proposition 5.1.** *For $\alpha < 0$, the wavefront set of the kernel $K$ of $\mathcal{G}^*\mathcal{G}$ satisfies,*

$$
WF'(K) \subset \Delta \cup C_1 \cup C_2,
$$

*where $\Delta$ is the diagonal in $T^*X \times T^*X$, $C_1$ is given by* (2.8) *and $C_2$ is defined as*

$$
\begin{aligned} C_2 = \Big\{ &(x,\xi;y,\xi') \,:\, \exists (s,t),\ (x,\xi) \in N^*(E(s,t)),\ (y,\xi') \in N^*(E(s,t)), \\ &\frac{\alpha}{A\widetilde{A}} + \frac{1}{B\widetilde{B}} = 0, (x_2,y_2) \neq (0,0) \Big\}, \end{aligned} \tag{5.1}
$$

*where $A = A(s,x)$ and $\widetilde{A} = A(s,y)$, $B = B(s,x)$ and $\widetilde{B} = B(s,y)$ and $E(s,t)$ is given by* (2.4)*. Furthermore, $\Delta$ and $C_1$ intersect cleanly in codimension 2, $\Delta$ and $C_2$ intersect cleanly in codimension 1, $C_2$ and $C_1$ intersect cleanly in codimension 1, and $\Delta \cap C_1 \cap C_2 = \emptyset$.*

*Proof.* In fact, this proposition is already proved in Proposition 4.1. Here, unlike the situation in Proposition 4.1, there is a nontrivial contribution to the wavefront of the composition from (4.11). Hence for $\alpha < 0$, we have that

$$
WF'(K) \subset \Delta \cup C_1 \cup C_2,
$$

To show no covector in $C_2$ has $x_2 = 0 = y_2$, one uses (4.11) and that $x_3 = 0 = y_3$ in (4.8). Finally, note that $\Delta \cap C_1 \cap C_2 = \emptyset$ since we exclude the points of intersection of $\Sigma_1$ and $\Sigma_2$ due to the cutoff function $g$ defined in Section 2.3. One can show that $C_2$ is an immersed conic Lagrangian manifold that is a two-sided fold using Definition 3.2 and the proof of Theorem 5.7 part (b)).

Using Def. 3.4 and the calculations above, one can also show that these manifolds intersect in the following ways:

(a) $\Delta$ intersects $C_1$ cleanly in codimension 2,

$$
\Delta \cap C_1 = \{(x,\xi;x,\xi) \in \Delta \,:\, x_2 = 0 = \xi_2\}.
$$

A proof is given in [1].

(b) $\Delta$ intersects $C_2$ cleanly in codimension 1,

$$
\begin{aligned} \Delta \cap C_2 &= \left\{ (x,\xi;x,\xi) \in \Delta \,:\, \frac{\alpha}{A^2} + \frac{1}{B^2} = 0 \right\} \\ &= \{(x,\xi;y,\eta) \in C_2 \,:\, x_1 = y_1,\, x_2 = y_2\}. \end{aligned}
$$

Note that the condition that $x_1 = y_1$ in $C_2$ implies $x_2 = \pm y_2$ and so the condition $x_2 = y_2$ does not increase the codimension of the intersection. Using [19], one can show the intersection is clean.

(c) $C_1$ intersects $C_2$ cleanly in codimension 1,

$$C_1 \cap C_2 = \left\{ (x, \xi; y, \eta) \in C_1 \,:\, \frac{\alpha}{A\tilde{A}} + \frac{1}{B\tilde{B}} = 0 \right\}.$$

Using [19], one can show the intersection is clean.

(d) $\Delta \cap C_1 \cap C_2 = \emptyset$ since we exclude the points of intersection of $\Sigma_1$ and $\Sigma_2$.

This completes the proof of the proposition. □

For the rest of the proof, we focus on $C_2$. Let $\beta = \sqrt{-\alpha}$, then $\beta > 1$. Let $(x, \xi, y, \xi') \in C_2$. then, by (5.1) there is an $(s, t)$ such that $x$ and $y$ are both in $E(s, t)$ and

$$\frac{\beta B}{A} = \frac{\widetilde{A}}{\beta \widetilde{B}}$$

where $A, \widetilde{A}, B, \widetilde{B}$ are given below (5.1). Therefore, if $(x, \xi, y, \xi') \in C_2$ then

$$\exists (s,t) \in (0,\infty)^2,\ \exists k \in (0,\infty), \quad x, y \in E(s,t) \quad \frac{\beta B(s,x)}{A(s,x)} = k, \quad \frac{\beta B(s,y)}{A(s,y)} = \frac{1}{k}. \tag{5.2}$$

A calculation shows that if $k \neq \beta$ then

$$\frac{\beta B(s,x)}{A(s,x)} = k \quad \Leftrightarrow \quad \left( x_1 - \frac{\beta^2 s(1+k^2)}{\beta^2 - k^2} \right)^2 + x_2^2 = \frac{\beta^2 s^2 k^2 (\beta^2+1)^2}{(\beta^2 - k^2)^2} - h^2 \tag{5.3}$$

If $k = \beta$, then $\beta B/A = k$ is the equation of a vertical line with $x_1$ intercept $(1-\beta^2)s/2$.

We first use this characterization of $C_2$ to prove Statements (1) and (3) of Theorem 2.6. As already mentioned, the diagonal relation $\Delta$ and $C_1$ given by (2.8) intersect cleanly in codimension 2 on either submanifold. Hence there is a well-defined $I^{p,l}$ class associated to $\Delta$ and $C_1$.

5.1. **Proof of Statement** (1) **of Theorem 2.6.** Recall from statement (1) of this theorem that the function $r_1$ is a cutoff function compactly supported in

$$O_1 = \{(s,t) \,:\, 0 < s < s_0 \text{ and } 0 < t < \infty\} \tag{5.4}$$

where $s_0$ (see (2.16)) can be written in terms of $\beta$ as

$$s_0 := \frac{h(\beta^2 - 1)}{\beta(\beta^2+1)}. \tag{5.5}$$

We show that for $(s,t) \in O_1$, there are no $x$ and $y$ satisfying (5.2). Therefore, $C_2 \cap WF'(K_1) = \emptyset$ where $K_1$ is the Schwartz kernel of $\mathcal{G}^* r_1 \mathcal{G}$.

So, assume for some $(s,t) \in O_1$ (5.2) holds. Then, the right-hand side of (5.3) can be estimated by

$$\frac{\beta^2 s^2 k^2 (\beta^2+1)^2}{(\beta^2-k^2)^2} - h^2 < h^2 \left( \frac{(\beta^2-1)^2 k^2 - (\beta^2-k^2)^2}{(\beta^2-k^2)^2} \right) = h^2 \left( \frac{(k^2-1)(\beta^4-k^2)}{(\beta^2-k^2)^2} \right).$$

Since $\beta > 1$, if $0 < k \le 1$, this calculation and (5.3) shows that $\frac{\beta B}{A} = k$ has no solution. Therefore there are no solutions to (5.2) if $0 < k \le 1$. Now assume for some $k > 1$, $\frac{\beta B(s,x)}{A(s,x)} = k$, then for (5.2) to have a solution that means that there must be a point $y \in E(s,t)$ with $\frac{\beta B(s,y)}{A(s,y)} = 1/k$. However, this is impossible since $0 < 1/k \le 1$. This shows that for $(s,t) \in O_1$, there is no solution to (5.2).

Now following the proof of Theorem 2.2, we achieve the result of Statement (1). □

5.2. **Proof of Statement** (3) **of Theorem 2.6.** Recall that the cutoff function $r_3(s,t)$ is compactly supported in

$$O_3 = \left\{(s,t) : s_0 < s < \infty \text{ and } t < t_s^- \text{ or } t > t_s^+\right\} \tag{5.6}$$

where $t_s^\pm$ is defined in (2.11). The operator we analyze in this part of the proof is $\mathcal{G}^* r_3 \mathcal{G}$.

We define the following set

$$C(s,k) := \left\{x \,:\, \frac{\beta B(s,x)}{A(s,x)} = k\right\}. \tag{5.7}$$

Note that $C(s,1) = \Sigma_{2,X}(s)$, $C(s,\beta)$ is the vertical line $x_1 = (1+\alpha)s/2 = (1-\beta^2)s/2$, and if $k$ is small enough, $C(s,k) = \emptyset$. By (5.3), when $k \neq \beta$ and $C(s,k) \neq \emptyset$ then $C(s,k)$ is the circle centered at $\left(\frac{\beta^2 s(1+k^2)}{\beta^2-k^2}, 0\right)$ and of radius

$$r(s,k) := \sqrt{\frac{\beta^2 s^2 k^2 (\beta^2+1)^2}{(\beta^2-k^2)^2} - h^2}. \tag{5.8}$$

Let $(s,t) \in O_3$. If there were a solution $(x,y)$ to (5.2) for some $k$, then $x \in E(s,t) \cap C(s,k)$ (as $\beta B/A = k$ on $C(s,k)$) *and* $y \in E(s,t) \cap C(s,1/k)$. If $t > t_s^+$ then the ellipse $E(s,t)$ encloses $\Sigma_{2,X}(s)$ by a calculation. Therefore, by the final statement of Lemma 5.2, $E(s,t)$ meets no circle $C(s,k)$ for $k \in (0,1]$ and so there is no solution to (5.2). Now, if $t < t_s^-$ then the ellipse $E(s,t)$ is enclosed by $\Sigma_{2,X}(s)$ and, by the final statement of Lemma 5.2, $E(s,t)$ meets no $C(s,k)$ for $k \in (1,\infty)$ and so there is no solution to (5.2) in this case, too. Therefore, $C_2 \cap WF'(K_3) = \emptyset$ where $K_3$ is the Schwartz kernel of $\mathcal{G}^* r_3 \mathcal{G}$. Now proceeding as in the proof of Theorem 2.2, we complete the proof of Statement (3) of Theorem 2.6. □

The rest of this section is devoted to the proof of Statement (2) of Theorem 2.6.

5.3. **Proof of Statement** (2) **of Theorem 2.6.** The reconstruction operator we consider in statement (2) of Theorem 2.6 is $\mathcal{G}^* r_2 \mathcal{G}$ where the mute $r_2$ has compact support in

$$O_2 = \{(s,t) : s_0 < s < \infty \ \text{ and } \ t_s^- < t < t_s^+\} \tag{5.9}$$

where $t_s^\pm$ is defined in (2.11) and where $s_0$ is defined by (5.5).

Recall that the canonical relation of $\mathcal{G}$ drops rank on the union of two sets, $\Sigma_1$ and $\Sigma_2$. Accordingly, we decompose $\mathcal{G}$ into components such that the canonical relation of each component is either supported near a subset of the union of these two sets, one of these two sets or away from both these sets. To do this, we define several cutoff functions.

5.3.1. *The primary cutoff functions $\psi_1$ and $\psi_2$.* The cutoff $\psi_1(x)$ will be equal to 1 near the $x_1$-axis and zero away from it, and $\psi_2(s,x)$ will be equal to one near $\Sigma_{2,X}(s)$ and equal to zero away from it as in Figure 1.

To define these functions precisely, we need to set up some preliminary relations. Because the mute function $r_2$ is zero near $s_0$ and has compact support, there is an $s_1 > s_0$ such that $r_2$ is zero for $s \leq s_1$ and all $t$. Because the radius $r(s_1,1) > 0$ and the function $r$ is continuous, there is a $k_1 \in (0,1)$ such that $r(s_1,k_1) > 0$. Since $r$ (see (5.8)) is an increasing function in $s$ and $k$ separately, we can choose $\epsilon > 0$ such that

$$r(s,k) \geq r(s_1,k_1) > 12\epsilon \ \text{ for } k \geq k_1, \text{ and } s \geq s_1. \tag{5.10}$$

Without loss of generality, we can assume

$$\epsilon < \frac{\min(\beta-1, 1-k_1, 1/4)}{6}. \tag{5.11}$$

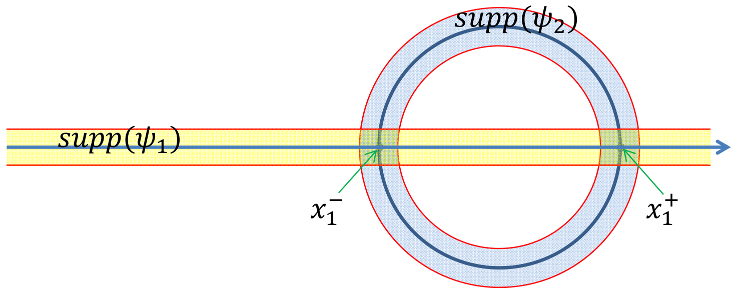


FIGURE 1. Picture of $\text{supp}(\psi_1)$ and $\text{supp}(\psi_2)$. Note that the middle circle is $\Sigma_{2,X}(s)$ and the circles are not exactly concentric.

Now, let $\psi_1$ be an infinitely differentiable function defined as follows:

$$\psi_1(x) = \begin{cases} 1, & |x_2| < \epsilon \\ 0, & |x_2| > 2\epsilon \end{cases} \tag{5.12}$$

and we extend this function smoothly between 0 and 1.

For $s > s_0$, let $k_0(s)$ be defined by

$$r(s, k_0(s)) = 0. \tag{5.13}$$

Note that $k_0(s)$ can be explicitly calculated using (5.8). So, if $k > k_0(s)$, $C(s,k)$ is a nontrivial circle. Finally, note that if $s \geq s_1$, then $k_1 > k_0(s)$; this is true because $r(s,k_1) \geq r(s_1,k_1) > 0$ for such $s$.

To define $\psi_2$ we first prove a lemma about the circles $C(s,k)$.

**Lemma 5.2.** *Let $s \geq s_1$.*

*(1) If $k > \beta$ then $C(s,k)$ is to the left of the vertical line $C(s,\beta)$ which is to the left of $C(s,\ell)$ for any $\ell \in (k_0(s),\beta)$.*

*(2) If $k_0(s) < j < k < \beta$ then $C(s,j)$ is contained inside $C(s,k)$, and these circles do not intersect.*

*(3) For any $\delta \in (0, 6\epsilon)$,*

$$\left\{ x : \left| \frac{\beta B(s,x)}{A(s,x)} \right| < \delta \right\} = \bigcup_{k\in I} C(s,k)$$

*is an open set containing $\Sigma_{2,X}(s) = C(s,1)$.*

*Proof.* Statement (1) of the lemma is a straightforward calculation.

Now, fix $s \geq s_1$. Let $k \in (k_0(s),\beta)$, then the endpoints of $C(s,k)$ on the $x_1$-axis are

$$x_\ell(k) = \frac{\beta^2 s(1+k^2)}{\beta^2 - k^2} - \sqrt{\frac{\beta^2 s^2 k^2 (\beta^2+1)^2}{(\beta^2-k^2)^2} - h^2}.$$
$$x_r(k) = \frac{\beta^2 s(1+k^2)}{\beta^2 - k^2} + \sqrt{\frac{\beta^2 s^2 k^2 (\beta^2+1)^2}{(\beta^2-k^2)^2} - h^2}$$

Clearly the functions $x_\ell$ and $x_r$ are smooth for $k \in (k_0(s),\beta)$. It is straightforward to see that $k \mapsto x_r(k)$ is a strictly increasing smooth function for $k \in (k_0(s),\beta)$.

We prove that the function $x_\ell$ is strictly decreasing by showing $x_\ell{}'$ is always negative. A somewhat tedious calculation shows that

$$x_\ell{}'(k) = \frac{\beta^2 s(\beta^2+1)k}{(\beta^2-k^2)^2}\left[2 - \frac{\frac{(\beta^2+1)s(\beta^2+k^2)}{\beta^2-k^2}}{\sqrt{\frac{\beta^2 s^2 k^2(\beta^2+1)^2}{(\beta^2-k^2)^2} - h^2}}\right]$$

By replacing the square root in this expression by the upper bound $\frac{\beta sk(\beta^2+1)}{(\beta^2-k^2)}$, we see that

$$x_\ell{}'(k) \leq \frac{\beta^2 s(\beta^2+1)k}{(\beta^2-k^2)^2\beta k}(-1)(\beta-k)^2$$

and the right-hand side of this expression is clearly negative.

The circles $C(s,\cdot)$ are symmetric about the $x_1$-axis, so if $j$ and $k$ are points in $(k_0(s),\beta)$ with $j<k$, since $x_\ell(k) < x_\ell(j) < x_r(j) < x_r(k)$, the circle $C(s,j)$ is strictly inside the circle $C(s,k)$. This proves (2).

By the choice of $\epsilon$ in (5.11), $1-6\epsilon > k_1$ and $1+6\epsilon < \beta$. Because $x_\ell(k)$ and $x_r(k)$ are smooth strictly monotonic functions with nonzero derivatives, $(1-6\epsilon, 1+6\epsilon) \ni k \mapsto C(s,k)$ is a foliation of an open, connected region containing $C(s,1) = \Sigma_{2,X}(s)$, and this proves (3). □

We define

$$\psi_2(s,x) = \begin{cases} 1 & \left|\frac{\beta B}{A} - 1\right| < \epsilon \\ 0 & \left|\frac{\beta B}{A} - 1\right| > 2\epsilon \end{cases} \tag{5.14}$$

and we extend smoothly between (which is possible by Lemma 5.2, statement (3)). By the lemma, $\psi_2(s,\cdot)$ is equal to 1 on an open neighborhood of $\Sigma_{2,X}(s)$ and zero away from $\Sigma_{2,X}(s)$.

We assume, without loss of generality, that $\psi_1$ and $\psi_2$ are symmetric about the $x_1$-axis.

*Remark* 5.3. We now can define the function $g(s,t)$ in Remark 2.5. We let

$$D(s,\epsilon) = \left\{(x_1,x_2)\ :\ |x_2| < \epsilon, \left|\frac{\beta B(s,x)}{A(s,x)} - 1\right| < \epsilon\right\}. \tag{5.15}$$

The set $D(s,4\epsilon)$ is represented by the shaded set in Figure 2 that is near $C(s,1) = \Sigma_{2,X}(s)$ and the $x_1$-axis. Let $g$ be a smooth function of $(s,t)$ that is zero if the ellipse $E(s,t)$ given in (2.4) intersects $D(s,4\epsilon)$ and is equal to 1 if $E(s,t)$ does not meet $D(s,5\epsilon)$.

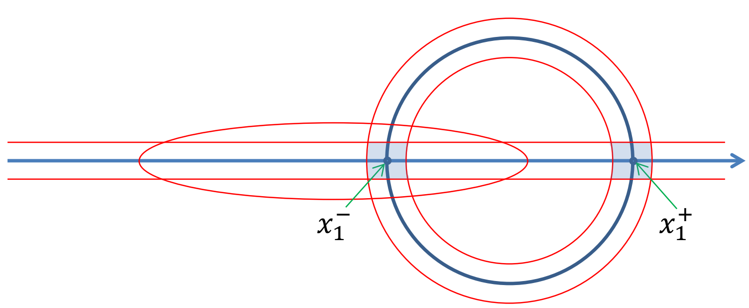


FIGURE 2. Picture of ellipse $E(s,t)$ that does not meet $D(s,4\epsilon)$. As discussed in Remark 2.5 and Remark 5.3, ellipses are muted by $g$ if they intersect $D(s,4\epsilon)$.

5.3.2. *Properties of $\mathcal{G}^*\mathcal{G}$ and end of proof.* We now write $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3$ where $\mathcal{G}_i$ are given in terms of their kernels

$$K_{\mathcal{G}_0} = \int e^{-i\varphi} a\psi_1\psi_2 d\omega, \quad K_{\mathcal{G}_1} = \int e^{-i\varphi} a\psi_1(1-\psi_2) d\omega,$$
$$K_{\mathcal{G}_2} = \int e^{-i\varphi} a(1-\psi_1)\psi_2 d\omega, \quad K_{\mathcal{G}_3} = \int e^{-i\varphi} a(1-\psi_1)(1-\psi_2) d\omega,$$

where $\varphi$ is the phase function of $\mathcal{G}$. The supports of the $\mathcal{G}_i$ are given in Figure 3.

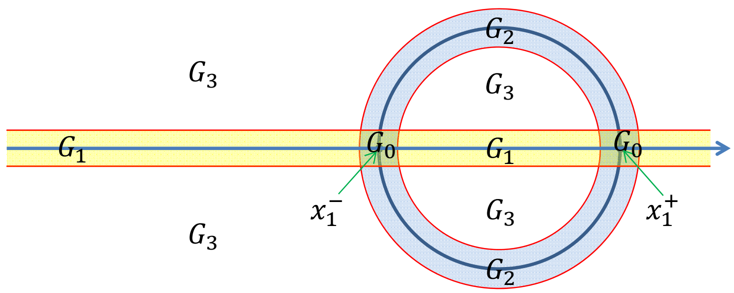


FIGURE 3. Picture indicating the rough locations of the support of $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$. Note that the circles are not exactly concentric.

Now we consider $\mathcal{G}^*\mathcal{G}$, which using the decomposition of $\mathcal{G}$ as above can be written as $\mathcal{G}^*\mathcal{G} = \mathcal{G}_0^*\mathcal{G} + (\mathcal{G}_1+\mathcal{G}_2)^*\mathcal{G}_0 + \mathcal{G}_1^*\mathcal{G}_1 + \mathcal{G}_2^*\mathcal{G}_2 + \mathcal{G}_1^*\mathcal{G}_2 + \mathcal{G}_2^*\mathcal{G}_1 + \mathcal{G}_1^*\mathcal{G}_3 + \mathcal{G}_2^*\mathcal{G}_3 + \mathcal{G}_3^*\mathcal{G}$

The theorem now follows from Lemmas 5.4-5.6, and Theorem 5.7, which we now state and prove. In the lemmas, we analyze the compositions above.

Recall that $\mathcal{G}_1$ and $\mathcal{G}_2$ are operators defined as follows:

$$\mathcal{G}_1 V(s,t) = \int e^{-\mathrm{i}\varphi(s,t,x,\omega)}\psi_1(x)(1-\psi_2(s,x))a(s,t,x,\omega)V(x)\mathrm{d}x\mathrm{d}\omega$$

and

$$\mathcal{G}_2 V(s,t) = \int e^{-\mathrm{i}\varphi(s,t,x,\omega)}(1-\psi_1(x))\psi_2(s,x)a(s,t,x,\omega)V(x)\mathrm{d}x\mathrm{d}\omega$$

**Lemma 5.4.** *The operators $\mathcal{G}_1^*\mathcal{G}_2$ and $\mathcal{G}_2^*\mathcal{G}_1$ are smoothing.*

*Proof.* We show that $\mathcal{G}_1^*\mathcal{G}_2$ is smoothing. The proof for the case of $\mathcal{G}_2^*\mathcal{G}_1$ is similar.

We have

$$\mathcal{G}_1^* V(x) = \int e^{\mathrm{i}\varphi(s,t,x,\omega)}\psi_1(x)(1-\psi_2(s,x))\overline{a(s,t,x,\omega)}V(s,t)\mathrm{d}s\mathrm{d}t\mathrm{d}\omega.$$

where $\psi_1(x)$ and $\psi_2$ are defined in (5.12) and (5.14) respectively. The Schwartz kernel of $\mathcal{G}_1^*\mathcal{G}_2$ is

$$K(x,y) = \int e^{i\omega(|y-\gamma_T(s)|+|y-\gamma_R(s)|-|x-\gamma_T(s)|-|x-\gamma_R(s)|)}\tilde{a}(x,y,s,\omega)\, ds d\omega,$$

where $\widetilde{a}(x,y,s,\omega)$ has the following products of cutoff functions as an additional factor:

$$g(s,t)\psi_1(x)(1-\psi_2(s,x))(1-\psi_1(y))\psi_2(y,s).$$

Here $t$ is determined from $s$ and $x$ as the value for which $x \in E(s,t)$. For this reason, in trying to understand the propagation of singularities, we need only to restrict ourselves, for each fixed $s \geq s_1$, to those base points $x$ and $y$ for which

$$x \in \operatorname{supp}(\psi_1(\cdot)(1-\psi_2(s,\cdot))),\ y \in \operatorname{supp}((1-\psi_1(\cdot))\psi_2(s,\cdot)). \tag{5.16}$$

We use this setup to show $\mathcal{G}_1^*\mathcal{G}_2$ is smoothing by showing its symbol is zero for covectors in $\mathcal{C}_\mathcal{G}^t \circ \mathcal{C}_\mathcal{G}$ (note that our argument shows that the symbol of the operator is zero in a neighborhood in $(T^*(X)\setminus \mathbf{0})^2$ of $\mathcal{C}_\mathcal{G}^t \circ \mathcal{C}_\mathcal{G}$). Let $(x,\xi,y,\xi') \in \mathcal{C}_\mathcal{G}^t \circ \mathcal{C}_\mathcal{G}$. Then, there is an $(s_2,t_2,\eta) \in T^*(Y)\setminus \mathbf{0}$ such that $(x,\xi,s_2,t_2,\eta) \in \mathcal{C}_\mathcal{G}^t$ and $(s_2,t_2,\eta,y,\xi') \in \mathcal{C}_\mathcal{G}$. For the rest of the proof, we fix this $s_2$. (If there are other values of $s$ associated to the composition, we repeat this proof for those values of $s$.)

Because $\mathcal{C}_\mathcal{G}^t \circ \mathcal{C}_\mathcal{G} \subset \Delta \cup C_1 \cup C_2$, we consider three cases separately.

I. **Covectors** $(x,\xi,y,\xi') \in \Delta \cap \left(\mathcal{C}_\mathcal{G}^t \circ \mathcal{C}_\mathcal{G}\right)$**:** In this case, $x=y$ and $x$ is in $\operatorname{supp}(\psi_1)\cap\operatorname{supp}(\psi_2) \subset D(s_2,4\epsilon)$. By the choice of the function $g(s,t)$ in Remark 5.3, the symbol of $\mathcal{G}_1^*\mathcal{G}_2$ is zero above such $x$.

II. **Covectors** $(x,\xi,y,\xi') \in C_1 \cap \left(\mathcal{C}_\mathcal{G}^t \circ \mathcal{C}_\mathcal{G}\right)$**:** In this case, $(x_1,x_2)=(y_1,-y_2)$ and the argument in case I shows that the symbol of $\mathcal{G}_1^*\mathcal{G}_2$ is zero for such $x$ and $y$

III. **Covectors** $(x,\xi,y,\xi') \in C_2 \cap \left(\mathcal{C}_\mathcal{G}^t \circ \mathcal{C}_\mathcal{G}\right)$**:** If $(x,\xi,y,\xi') \in C_2 \cap \left(\mathcal{C}_\mathcal{G}^t \circ \mathcal{C}_\mathcal{G}\right)$, then for some $(s_2,t_2)$ above, there is a $k_2 > k_0(s_2)$, such that

$$x \in E(s_2,t_2)\cap\operatorname{supp}(\psi_1)\cap\operatorname{supp}(1-\psi_2(s,\cdot))\cap C(s_2,k_2) \tag{5.17}$$
$$y \in E(s_2,t_2)\cap\operatorname{supp}(1-\psi_1)\cap\operatorname{supp}(\psi_2(s,\cdot))\cap C\left(s_2,1/k_2\right). \tag{5.18}$$

Using (5.17), the fact that $k_2=\beta B(s_2,x)/A(s_2,x)$, we see that $|x_2|<2\epsilon$ and $|1-k_2|>\epsilon$. Now, using the restriction on $1/k_2$ in (5.18) and the fact that $\epsilon<1/4$, we see $|1-k_2|<4\epsilon$. Putting this together shows that

$$1-4\epsilon < k = \frac{\beta B(s_2,x)}{A(s_2,x)} < 1+4\epsilon.$$

Since $|x_2|<2\epsilon$, this shows that $x\in D(s_2,4\epsilon)$. Therefore $E(s_2,t_2)\cap D(s_2,4\epsilon)\neq\emptyset$ and $g(s_2,t_2)=0$ by Remark 5.3. Therefore, the symbol of $\mathcal{G}_1^*\mathcal{G}_2$ is zero near $(x,\xi,y,\xi')$ so $\mathcal{G}_1^*\mathcal{G}_2$ is smoothing near $(x,\xi,y,\xi')$.

This finishes the proof that $\mathcal{G}_1^*\mathcal{G}_2$ is smoothing. □

**Lemma 5.5.** *The operator $\mathcal{G}_0$ is smoothing.*

*Proof.* Recall that the Schwartz kernel of $\mathcal{G}_0$ is

$$K_{\mathcal{G}_0} = \int e^{-i\varphi} a\psi_1(x)\psi_2(s,x)d\omega.$$

For each fixed $s\geq s_1$, the support of $\psi_1(\cdot)\psi_2(s,\cdot)$ is inside $D(s,4\epsilon)$ and by the choice of the function $g(s,t)$ in Remark 5.3, the symbol of $\mathcal{G}_0$ is zero above such $(s,x)$. □

**Lemma 5.6.** *The operators $\mathcal{G}_1^*\mathcal{G}_3$, $\mathcal{G}_2^*\mathcal{G}_3$ and $\mathcal{G}_3^*\mathcal{G}$ can be decomposed as a sum of operators belonging to the space $I^3(\Delta\setminus(C_1\cup C_2)) + I^3(C_1\setminus(\Delta\cup C_2)) + I^3(C_2\setminus(\Delta\cup C_1))$.*

*Proof.* Each of these compositions is covered by the transverse intersection calculus.

We decompose $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ into a sum of operators on which the compositions will be easier to analyze. This is represented in Figure 4.

For $\mathcal{G}_1$, note that $\Sigma_{2,X}(s)$ divides $\{|x_2|<2\epsilon\}$ in three regions since $r(s,1)>12\epsilon$ by (5.10). Let $H_1(s)$ be the part of $\{|x_2|<2\epsilon\}\setminus\Sigma_{2,X}(s)$ to the left of $\Sigma_{2,X}(s)$ and let $H_2(s)$ be the part inside $\Sigma_{2,X}(s)$ and $H_3(s)$ the part to the right of $\Sigma_{2,X}(s)$. Define our partitioned operators as follows $\mathcal{G}_1 = \mathcal{G}_1^1+\mathcal{G}_1^2+\mathcal{G}_1^3$ where

$$\mathcal{G}_1^i V(s,t) = \int e^{-i\varphi(s,t,x,\omega)}\psi_1(x)(1-\psi_2(s,x))\chi_{H_i(s)}(x)a(s,t,x,\omega)V(x)dxd\omega$$

for $i=1,2,3$. Note that the symbols are all smooth because $\chi_{H_j(s)}(x)\psi_1(x)\left(1-\psi_2(s,x)\right)$ is a smooth cutoff function in $(s,x)$ since the support of $\psi_1$ is inside $\{|x_2|<2\epsilon\}$ and the support of $(1-\psi_2(s,\cdot))$ does not meet $\Sigma_2(s)$.

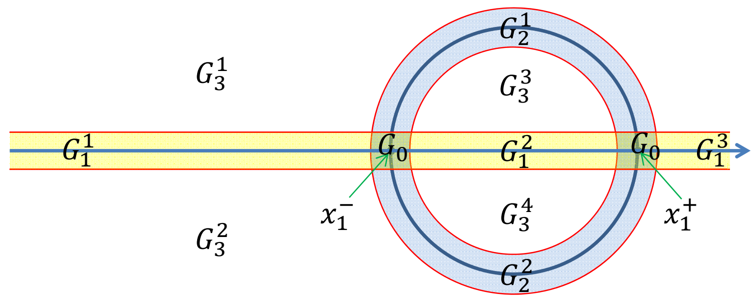


FIGURE 4. Picture indicating the rough locations of the support of $\mathcal{G}_0$, and the partitioned functions $\mathcal{G}_1^i$, $\mathcal{G}_2^j$, and $\mathcal{G}_3^k$. Note that the circles are not exactly concentric.

We decompose $\mathcal{G}_2$ into two operators in a similar way. Let $I_1$ be the open upper half plane and let $I_2$ be the open lower half plane. Define

$$\mathcal{G}_2^i V(s,t) = \int e^{-i\varphi(s,t,x,\omega)}(1-\psi_1(x))\psi_2(s,x)\chi_{I_j(s)}(x)a(s,t,x,\omega)V(x)dxd\omega \tag{5.19}$$

for $j = 1, 2$ Because the functions $(1 - \psi_1(\cdot))\psi_2(s,\cdot)$ are supported away from the $x_1$ axis, these symbols are smooth.

We decompose $\mathcal{G}_3$ into four operators in a similar way using Figure 4: $\Sigma_{2,X}(s)$ divides $\{x_2 \neq 0\}$ into four regions $J_1(s)$, the unbounded region above the $x_1$-axis, $J_2(s)$, it's mirror image in the $x_1$-axis, $J_3(s)$, the bounded region inside $\Sigma_{2,X}(s)$ and above the $x_1$-axis, and its mirror image, $J_4(s)$. We define

$$\mathcal{G}_3^k V(s,t) = \int e^{-i\varphi(s,t,x,\omega)}(1-\psi_1(x))(1-\psi_2(s,x))\chi_{J_k(s)}(x)a(s,t,x,\omega)V(x)dxd\omega$$

for $k = 1, 2, 3, 4$, and because of the cutoffs used, these are all FIO with smooth symbols.

To find the canonical relation of ${\mathcal{G}_1^j}^*\mathcal{G}_3^k$, we consider $(x,\xi,y,\xi') \in \mathcal{C}_\mathcal{G}^t \circ \mathcal{C}_\mathcal{G}$ and let $(s,t) \in Y$ such that $(x,\xi,s,t,\eta) \in \mathcal{C}_\mathcal{G}^t$ and $(s,t,\eta,y,\xi') \in \mathcal{C}_\mathcal{G}$. In any case, $(\mathcal{G}_1^i)^*\mathcal{G}_3^j$ has canonical relation a subset of $\mathcal{C}_\mathcal{G}^t \circ \mathcal{C}_\mathcal{G} \subset \Delta \cup C_1 \cup C_2$. To find which subset, we consider the restriction that the supports of the $\mathcal{G}_i^j$ put on $x$ and $y$. We use the fact that $x$ and $y$ are on $E(s,t)$ plus the following rules to understand the canonical relations of these operators:

$(i)$ If the supports exclude $x$ and $y$ from being equal, then the canonical relation $(WF')$ of the composed operator does not include $\Delta$.
$(ii)$ if the supports exclude $x$ and $y$ from being reflections in the $x_1$ axis then the canonical relation of the composed operator does not include $C_1$.
$(iii)$ If the supports exclude $x$ from being outside $\Sigma_{2,X}(s)$ and $y$ being inside or vice versa, then the canonical relation of the composed operator does not include $C_2$.

We first consider $(\mathcal{G}_1)^*\mathcal{G}_3$. To do this, we partition $\mathcal{G}_1$ further. Let $u$ be a smooth cutoff function supported in $[-\epsilon,\epsilon]$ and equal to one on $[-\epsilon/2,\epsilon/2]$ and let $\sigma^+ = \chi_{[0,2\epsilon]}(1-u)\psi_1(1-\psi_2)$, $\sigma^o = \chi_{[-\epsilon,\epsilon]}u\psi_1(1-\psi_2)$, and $\sigma^- = \chi_{[-2\epsilon,0]}(1-u)\psi_1(1-\psi_2)$ where the characteristic functions and $u$ are functions of $x_2$. Note that, for each fixed $s$ and functions of $x$, $\text{supp}(\sigma^+) \subset \mathbb{R} \times [\epsilon/2, 2\epsilon]$, $\text{supp}(\sigma^o) \subset \mathbb{R} \times [-\epsilon,\epsilon]$, $\text{supp}(\sigma^-) \subset \mathbb{R} \times [-2\epsilon, -\epsilon/2]$. All these functions are smooth and $\psi_1(1-\psi_2) = \sigma^+ + \sigma^o + \sigma^-$. This allows us to divide up each $\mathcal{G}_1^j$ $(j = 1, 2, 3)$ into the sum of three operators where $\mathcal{G}_1^{j+}(V)$ has symbol equal to the symbol of $\mathcal{G}$ but multiplied by $\sigma^+ H_j$, $\mathcal{G}_1^{jo}(V)$ has symbol equal to the symbol of $\mathcal{G}$ but multiplied by $\sigma^{o}\, H_j$, and $\mathcal{G}_1^{j-}(V)$ has symbol equal to the symbol of $\mathcal{G}$ but multiplied by $\sigma^{-}\, H_j$. Note that $\mathcal{G}_1^{j} = \mathcal{G}_1^{j+} + \mathcal{G}_1^{jo} + \mathcal{G}_1^{j-}$.

We now analyze the composition $\mathcal{G}_1^{1*}\mathcal{G}_3^1$ using this partition of $\mathcal{G}_1^1$. Consider the composition $(\mathcal{G}_1^{1+})^*\mathcal{G}_3^1$. Because both operators are supported in $x$ above the $x_1$ axis, the canonical relation of this composition cannot intersect $C_1$ (see $(ii)$). Because they are both supported outside $\Sigma_{2,X}(s)$, it cannot intersect $C_2$ (since $C_2$ associates points inside $\Sigma_{2,X}(s)$ only with points outside $\Sigma_{2,X}(s)$ and vice versa by $(iii)$). So this shows $(\mathcal{G}_1^{1+})^*\mathcal{G}_3^1 \in I(\Delta \setminus C_1)$.

Note that we use the transverse intersection calculus to show $(\mathcal{G}_1^{1+})^*\mathcal{G}_3^1$ and each of the other operators in this lemma are regular FIO.

Now, we consider $(\mathcal{G}_1^{1o})^*\mathcal{G}_3^1$. Note that $\mathcal{G}_1^{1o}$ is supported in $x$ in $|x_2| < \epsilon$ and $\mathcal{G}_3^1$ is supported in $x_2 > \epsilon$. Therefore, the canonical relation of the composition can include neither $\Delta$ nor $C_1$ by $(i)$, $(ii)$. Furthermore, because they are both supported outside $\Sigma_{2,X}(s)$, it does not contain $C_2$ by $(iii)$. Therefore, $(\mathcal{G}_1^{1o})^*\mathcal{G}_3^1$ is smoothing.

Next, we consider $(\mathcal{G}_1^{1-})^*\mathcal{G}_3^1$. The argument is similar to the case $(\mathcal{G}_1^{1+})^*\mathcal{G}_3^1$, but this canonical relation is contained in $C_1 \setminus \Delta$.

This shows that $(\mathcal{G}_1^1)^*\mathcal{G}_3^1$ is a sum of operators in $I^3(\Delta \setminus (C_1 \cup C_2)) + I^3(C_1 \setminus (\Delta \cup C_2))$.

The proof that $(\mathcal{G}_1^1)^*\mathcal{G}_3^2 \in I^3(\Delta \setminus (C_1 \cup C_2)) + I^3(C_1 \setminus (\Delta \cup C_2))$ follows using the same arguments but the roles of $\mathcal{G}_1^{1-}$ and $\mathcal{G}_1^{1+}$ are switched because $\mathcal{G}_3^2$ has support in $x$ below the $x_1$-axis and below $\Sigma_{2,X}(s)$.

Now we consider $(\mathcal{G}_1^1)^*\mathcal{G}_3^3$. Because the support in $x$ of $\mathcal{G}_1^1$ is to the left of $\Sigma_{2,X}(s)$ and the support of $\mathcal{G}_3^3$ is inside, the canonical relation of $(\mathcal{G}_1^1)^*\mathcal{G}_3^3$ cannot intersect $\Delta$ (since there are no points $(x, \xi, x, \xi)$ in that canonical relation by the support condition and $(i)$ and it cannot intersect $C_1$ for a similar reason by $(ii)$. So $(\mathcal{G}_1^1)^*\mathcal{G}_3^3 \in I^3(C_2 \setminus (\Delta \cup C_1))$.

A similar argument using symmetry of support of $\mathcal{G}_3^3$ and $\mathcal{G}_3^4$ in the $x_1$ axis shows that $(\mathcal{G}_1^1)^*\mathcal{G}_3^4 \in I^3(C_2 \setminus (\Delta \cup C_1))$.

Putting these together, we see that $(\mathcal{G}_1^1)^*\mathcal{G}_3 \in I^3(\Delta \setminus (C_1 \cup C_2)) + I^3(C_1 \setminus (\Delta \cup C_2)) + I^3(C_2 \setminus (\Delta \cup C_1))$.

The proof that $(\mathcal{G}_1^2)^*\mathcal{G}_3 \in I^3(\Delta \setminus (C_1 \cup C_2)) + I^3(C_1 \setminus (\Delta \cup C_2)) + I^3(C_2 \setminus (\Delta \cup C_1))$ is similar but here we use the partition of $\mathcal{G}_1^2$: $\mathcal{G}_1^{2+}$, $\mathcal{G}_1^{2o}$ and $\mathcal{G}_1^{2-}$. In a similar way, $(\mathcal{G}_1^3)^*\mathcal{G}_3 \in I^3(\Delta \setminus (C_1 \cup C_2)) + I^3(C_1 \setminus (\Delta \cup C_2)) + I^3(C_2 \setminus (\Delta \cup C_1))$.

Thus, $(\mathcal{G}_1)^*\mathcal{G}_3 \in I^3(\Delta \setminus (C_1 \cup C_2)) + I^3(C_1 \setminus (\Delta \cup C_2) + I^3(C_2 \setminus (\Delta \cup C_1))$.

Now we consider $(\mathcal{G}_2)^*\mathcal{G}_3$. Here we partition $\mathcal{G}_2^j$, $j = 1, 2$ into three operators with smooth symbols as we did for $\mathcal{G}_1$:

- $\mathcal{G}_2^{j+}$ will have support in $x$ for fixed $s$ in the union of circles $\cup_{k \in [1+\epsilon, 1+2\epsilon]} C(s, k)$ (outside of $\Sigma_{2,X}(s)$),
- $\mathcal{G}_2^{jo}$ will have support in $x$ for fixed $s$ in the union of circles $\cup_{k \in [1-\epsilon, 1+\epsilon]} C(s, k)$ (surrounding $\Sigma_{2,X}(s)$) and be equal to the symbol of $\mathcal{G}_2$ in $\cup_{k \in [1-\epsilon/2, 1+\epsilon/2]} C(s, k)$, and
- $\mathcal{G}_2^{j-}$ will have support in $x$ for fixed $s$ in the union of circles $\cup_{k \in [1-2\epsilon, 1-\epsilon]} C(s, k)$ (inside $\Sigma_{2,X}(s)$).

The proof follows similar arguments as for $(\mathcal{G}_1)^*\mathcal{G}_3$ and it shows $(\mathcal{G}_2)^*\mathcal{G}_3 \in I^3(\Delta \setminus (C_1 \cup C_2)) + I^3(C_1 \setminus (\Delta \cup C_2) + I^3(C_2 \setminus (\Delta \cup C_1))$.

Finally, we consider $(\mathcal{G}_3)^*\mathcal{G}$. By symmetry of the conditions $(i)$, $(ii)$, $(iii)$, we justify $(\mathcal{G}_3)^*\mathcal{G}_1$ and $(\mathcal{G}_3)^*\mathcal{G}_2$ are in $I^3(\Delta \setminus (C_1 \cup C_2)) + I^3(C_1 \setminus (\Delta \cup C_2)) + I^3(C_2 \setminus (\Delta \cup C_1))$. So, the only composition to consider is $(\mathcal{G}_3)^*\mathcal{G}_3$, and by analyzing all combinations, we see $(\mathcal{G}_3)^*\mathcal{G}_3 \in I^3(\Delta \setminus (C_1 \cup C_2)) + I^3(C_1 \setminus (\Delta \cup C_2)) + I^3(C_2 \setminus (\Delta \cup C_1))$. This finishes the proof.

□

We are left with the analysis of the compositions $\mathcal{G}_1^*\mathcal{G}_1$ and $\mathcal{G}_2^*\mathcal{G}_2$. This is the content of the next theorem:

**Theorem 5.7.** *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be as above. Then*

*(a)* $\mathcal{G}_1^*\mathcal{G}_1 \in I^{3,0}(\Delta, C_1) + I^3(C_2 \setminus (\Delta \cup C_1))$.
*(b)* $\mathcal{G}_2^*\mathcal{G}_2 \in I^{3,0}(\Delta, C_2) + I^{3,0}(C_1, C_2)$.

*Proof.* Consider the intersections of $\Delta$, $C_1$, $C_2$. We have that $\Delta$ intersects $C_1$ cleanly in codimension 2; $\Delta$ intersects $C_2$ cleanly in codimension 1 and $C_1$ intersects $C_2$ cleanly in codimension 2.

For part (a) we decompose $\mathcal{G}_1 = \mathcal{G}_1^1 + \mathcal{G}_1^2 + \mathcal{G}_1^3$. Now, we consider the compositions that $(\mathcal{G}_1^j)^*\mathcal{G}_1^j$ for $j = 1, 2, 3$. Using (i), (ii), and (iii), we have that $WF'((\mathcal{G}_1^j)^*\mathcal{G}_1^j) \subset \Delta \cup C_1$. Then, using a proof similar to the one for Theorem 2.2, we see that $(\mathcal{G}_1^j)^*\mathcal{G}_1^j \in I^{3,0}(\Delta, C_1)$.

Arguments using (i), (ii), and (iii) show that the cross terms $(\mathcal{G}_1^1)^*\mathcal{G}_1^2$, $(\mathcal{G}_1^2)^*\mathcal{G}_1^1$, $(\mathcal{G}_1^2)^*\mathcal{G}_1^3$, and $(\mathcal{G}_1^3)^*\mathcal{G}_1^2$ are in $I^3(C_2 \setminus (\Delta \cup C_1))$ and $(\mathcal{G}_1^3)^*\mathcal{G}_1^1$ and $(\mathcal{G}_1^1)^*\mathcal{G}_1^3$ are smoothing.

Now, we consider part (b) and the operator $\mathcal{G}_2^*\mathcal{G}_2$.

We recall that $\Sigma_1$ and $\Sigma_2$ are disjoint, $\Sigma_2 \in C \setminus \Sigma_1$ thus $C \setminus \Sigma_1$ is a two sided fold. Next we use [19] to get that $(C \setminus \Sigma_1)^t \circ (C \setminus \Sigma_1) = \Delta \cup C_2$, and that $C_2$ is a two sided fold.

We use the decomposition (5.19) $\mathcal{G}_2 = \mathcal{G}_2^1 + \mathcal{G}_2^2$ where $\mathcal{G}_2^1$ is supported in the upper part of $\Sigma_2$ and $\mathcal{G}_2^2$ is supported in the lower part of $\Sigma_2$. Note that the support in $x$ of $\mathcal{G}_2^1$ and $\mathcal{G}_2^2$ are disjoint.

Then using Theorem 3.9 we have that

$$(\mathcal{G}_2^1)^*\mathcal{G}_2^1 \in I^{3,0}(\Delta, C_2) \text{ and } (\mathcal{G}_2^2)^*\mathcal{G}_2^2 \in I^{3,0}(\Delta, C_2).$$

Consider the operator $R$ defined as follows:

$$RV(x_1, x_2) = V(x_1, -x_2).$$

This is a Fourier integral operator of order 0 and it is easy to check its canonical relation is $C_1$. Let $\hat{\mathcal{G}} = \mathcal{G}_2^2 \circ R$. We have $\hat{\mathcal{G}}^*\mathcal{G}_2^1 \in I^{3,0}(\Delta, C_2)$. Note that $C_1 \circ \Delta = C_1$, $C_1 \circ C_2 = C_2$ and $C_1 \times \Delta$ (as well as $C_1 \times C_2$) intersects $T^*X \times \Delta_{T^*X} \times T^*X$ transversally. Using [16, Proposition 4.1], this implies that $R^*\tilde{\mathcal{G}}^*\mathcal{G}_2^1 \in I^{3,0}(C_1, C_2)$. Since $\tilde{\mathcal{G}}^* = R^*(\mathcal{G}_2^2)^*$ and $(R^*)^2 = \mathrm{Id}$ we have $(\mathcal{G}_2^2)^*\mathcal{G}_2^1 \in I^{3,0}(C_1, C_2)$

Similarly, we show that

$$(\mathcal{G}_2^1)^*\mathcal{G}_2^2 \in I^{3,0}(C_1, C_2).$$

This concludes the proof of Statement (2) of Theorem 2.6. □

5.4. **Beam forming.** This is equivalent to assuming the scatterer $V$ has support in either the open half-plane $x_2 > 0$ or $x_2 < 0$. In this case, $C_1$ does not appear in the analysis.

**Theorem 5.8.** *Let $\mathcal{G}$ be as in* 2.11 *of order $\frac{3}{2}$. Assume the amplitude of $\mathcal{G}$ is nonzero only on a subset of either the upper half-plane ($x_2 > 0$) or the lower half plane $x_2 < 0$ and bounded away from the $x_1$ axis. Then $\mathcal{G}^*\mathcal{G} \in I^{3,0}(\Delta, C_2)$, where $C_2$ is given by* 5.1.

*Proof.* We assume $x_2 > 0$ (the other case is similar), $\Sigma_1$ is empty and $\pi_L$ and $\pi_R$ have fold singularities along $\Sigma_2$ as proved in Proposition 5.1. Thus $C^t \circ C = \Delta \cup C_2$ where $C_2$ is a two-sided fold. Using the results in Felea [3] and Nolan [21], we have that $\mathcal{G}^*\mathcal{G} \in I^{3,0}(\Delta, C_2)$. □

In this case, $C_2$ does affect the added singularities and a statement similar to Remark 2.3 holds but with $C_2$ replacing $C_1$.

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## Appendix A. Proofs of iterated regularity for $\mathcal{F}$ $(\alpha \geq 0)$

In this section, we prove that each of the $\tilde{p}_i$ given in (4.12) is a sum of products of derivatives of $\Phi$ and smooth functions. This will finish the proof that $\mathcal{F}^*\mathcal{F} \in I^{3,0}(\Delta, C_1)$.

A.1. **Expression for $x_1 - y_1$.** We will use the same prolate spheroidal coordinates 4.8 with foci $\gamma_R(s)$ and $\gamma_T(s)$ to solve for $x$ and $y$. We have

$$\begin{aligned} x_1 - y_1 &= \left(\frac{1+\alpha}{2}s + \frac{1-\alpha}{2}s\cosh\rho\cos\phi\right) \\ &\quad - \left(\frac{1+\alpha}{2}s + \frac{1-\alpha}{2}s\cosh\rho'\cos\phi'\right) \\ &= \frac{1-\alpha}{2}s\left(\cosh\rho\cos\phi - \cosh\rho'\cos\phi'\right) \\ &= \frac{1-\alpha}{2}s\left((\cosh\rho - \cosh\rho')\cos\phi + \cosh\rho'(\cos\phi - \cos\phi')\right). \end{aligned} \tag{A.1}$$

We have

$$\begin{aligned} \partial_\omega\Phi &= \left(\|y-\gamma_T(s)\| + \|y-\gamma_R(s)\| - (\|x-\gamma_T(s)\| + \|x-\gamma_R(s)\|)\right) \\ &= (1-\alpha)s(\cosh\rho' - \cosh\rho). \end{aligned}$$

Therefore in (A.1), it is enough to express $\cos\phi - \cos\phi'$ in terms of $\partial_\omega\Phi$ and $\partial_s\Phi$. We obtain:

$$\begin{aligned} \frac{\partial_s\Phi}{\omega} &= \left(\alpha\frac{x_1-\alpha s}{A} + \frac{x_1 - s}{B}\right) - \left(\alpha\frac{y_1-\alpha s}{A'} + \frac{y_1-s}{B'}\right) \\ &= \alpha\frac{\cosh\rho\cos\phi + 1}{\cosh\rho + \cos\phi} + \frac{\cosh\rho\cos\phi - 1}{\cosh\rho - \cos\phi} \\ &\quad - \left(\alpha\frac{\cosh\rho'\cos\phi' + 1}{\cosh\rho' + \cos\phi'} + \frac{\cosh\rho'\cos\phi' - 1}{\cosh\rho' - \cos\phi'}\right) \end{aligned}$$

Combining the first and the third term, and second and the fourth term above and then simplifying, we get

$$\begin{aligned}
&= \alpha \frac{(\cos\phi - \cos\phi')(\cosh\rho\cosh\rho' - 1) + (\cosh\rho - \cosh\rho')(\cos\phi\cos\phi' - 1)}{(\cosh\rho' + \cos\phi')(\cosh\rho + \cos\phi)} \\
&\quad + \frac{(\cos\phi - \cos\phi')(\cosh\rho\cosh\rho' - 1) + (\cosh\rho - \cosh\rho')(1 - \cos\phi\cos\phi')}{(\cosh\rho - \cos\phi)(\cosh\rho' - \cos\phi')} \\
&= (\cos\phi - \cos\phi')(\cosh\rho\cosh\rho' - 1)\left(\frac{\alpha}{(\cosh\rho' + \cos\phi')(\cosh\rho + \cos\phi)} + \frac{1}{(\cosh\rho - \cos\phi)(\cosh\rho' - \cos\phi')}\right) \\
&\quad + (\cosh\rho - \cosh\rho')(\cos\phi\cos\phi' - 1)\left(\frac{\alpha}{(\cosh\rho' + \cos\phi')(\cosh\rho + \cos\phi)} - \frac{1}{(\cosh\rho - \cos\phi)(\cosh\rho' - \cos\phi')}\right)
\end{aligned}$$

Now denote

$$P_{\pm} := \frac{\alpha}{(\cosh\rho' + \cos\phi')(\cosh\rho + \cos\phi)} \pm \frac{1}{(\cosh\rho - \cos\phi)(\cosh\rho' - \cos\phi')}$$

Note that since $\alpha > 0$, $P_+ > 0$. Therefore we have

$$\cos\phi - \cos\phi' = \frac{1}{(\cosh\rho\cosh\rho' - 1)P_+}\left(\frac{1}{\omega}\partial_s\Phi - \frac{(1 - \cos\phi\cos\phi')}{(1-\alpha)s}P_-\partial_\omega\Phi\right)$$

Now using this expression for the difference of cosines in (A.1), we are done.

A.2. **Expression for $x_2^2 - y_2^2$.** We have

$$\begin{aligned}
x_2^2 - y_2^2 &= \frac{(1-\alpha)^2 s^2}{4}\left(\sinh^2\rho\sin^2\phi\cos^2\theta - \sinh^2\rho'\sin^2\phi'\cos^2\theta'\right) \\
&= \frac{(1-\alpha)^2 s^2}{4}\left(\sinh^2\rho\sin^2\phi - \sinh^2\rho'\sin^2\phi'\right) \\
&\quad + \frac{(1-\alpha)^2 s^2}{4}\left(-\sinh^2\rho\sin^2\phi\sin^2\theta + \sinh^2\rho'\sin^2\phi'\sin^2\theta'\right) \qquad \text{(A.2)}
\end{aligned}$$

Since $x_3 = y_3 = 0$, we have that the last term in (A.2) is 0.

Now we can write $\sinh^2\rho\sin^2\phi - \sinh^2\rho'\sin^2\phi' = (\cosh^2\rho - \cosh^2\rho')\sin^2\phi - (\cos^2\phi - \cos^2\phi')\sinh^2\rho' = (\cosh\rho - \cosh\rho')(\cosh\rho + \cosh\rho')\sin^2\phi - (\cos\phi - \cos\phi')(\cos\phi + \cos\phi')\sinh^2\rho'$.
Since $\cosh\rho - \cosh\rho'$ and $\cos\phi - \cos\phi'$ can be expressed in terms of $\partial_\omega\Phi$ and $\partial_s\Phi$ as above, we are done.

A.3. **Expression for $\xi_1 - \eta_1$.** We have

$$\xi_1 = -\omega\left(\frac{x_1 - \alpha s}{\sqrt{(x_1 - \alpha s)^2 + x_2^2 + h^2}} + \frac{x_1 - s}{\sqrt{(x_1 - s)^2 + x_2^2 + h^2}}\right)$$

$$\eta_1 = -\omega\left(\frac{y_1 - \alpha s}{\sqrt{(y_1 - \alpha s)^2 + y_2^2 + h^2}} + \frac{y_1 - s}{\sqrt{(y_1 - s)^2 + y_2^2 + h^2}}\right)$$

In prolate spheroidal coordinates, we have

$$\begin{aligned}
\frac{\xi_1-\eta_1}{2\omega} &= \left(\frac{\sinh^2\rho'\cos\phi'}{\cosh^2\rho'-\cos^2\phi'} - \frac{\sinh^2\rho\cos\phi}{\cosh^2\rho-\cos^2\phi}\right) \\
&= \left(\frac{\sinh^2\rho'\cos\phi'}{\cosh^2\rho'-\cos^2\phi'} - \frac{\sinh^2\rho'\cos\phi'}{\cosh^2\rho-\cos^2\phi}\right. \\
&\qquad \left. + \frac{\sinh^2\rho'\cos\phi'}{\cosh^2\rho-\cos^2\phi} - \frac{\sinh^2\rho\cos\phi}{\cosh^2\rho-\cos^2\phi}\right) \\
&= \sinh^2\rho'\cos\phi'\left(\frac{\cosh^2\rho-\cosh^2\phi'+\cos^2\phi'-\cos^2\phi}{(\cosh^2\rho'-\cos^2\phi')(\cosh^2\rho-\cos^2\phi)}\right) \\
&\qquad + \left(\frac{\sinh^2\rho'\cos\phi'-\sinh^2\rho\cos\phi}{\cosh^2\rho-\cos^2\phi}\right). \\
&= \sinh^2\rho'\cos\phi'\left(\frac{\cosh^2\rho-\cosh^2\phi'+\cos^2\phi'-\cos^2\phi}{(\cosh^2\rho'-\cos^2\phi')(\cosh^2\rho-\cos^2\phi)}\right) \\
&\qquad + \left(\frac{(\cosh^2\rho'-\cosh^2\rho)\cos\phi'+\sinh^2\rho(\cos\phi'-\cos\phi)}{\cosh^2\rho-\cos^2\phi}\right).
\end{aligned}$$

As before we get the terms $\cosh\rho-\cosh\rho'$ and $\cos\phi-\cos\phi'$ which can be expressed in terms of $\partial_\omega\Phi$ and $\partial_s\Phi$.

A.4. **Expression for $(x_2-y_2)(\xi_2+\eta_2)$.** We have
$\xi_2 = -\omega\left(\frac{x_2}{\sqrt{(x_1-\alpha s)^2+x_2^2+h^2}} + \frac{x_2}{\sqrt{(x_1-s)^2+x_2^2+h^2}}\right)$ and
$\eta_2 = -\omega\left(\frac{y_2}{\sqrt{(y_1-\alpha s)^2+y_2^2+h^2}} + \frac{y_2}{\sqrt{(y_1-s)^2+y_2^2+h^2}}\right)$
Thus

$$\begin{aligned}
-\frac{(x_2-y_2)(\xi_2+\eta_2)}{\frac{4}{(1-\alpha)s}\omega} &= \frac{x_2^2\cosh\rho}{\cosh^2\rho-\cos^2\phi} - \frac{y_2^2\cosh\rho'}{\cosh^2\rho'-\cos^2\phi'} \\
&\qquad + x_2y_2\left(\frac{\cosh\rho'}{\cosh^2\rho'-\cos^2\phi'} - \frac{\cosh\rho}{\cosh^2\rho-\cos^2\phi}\right) \\
&= (x_2^2-y_2^2)\frac{\cosh\rho}{\cosh^2\rho-\cos^2\phi} \\
&\qquad - y_2(x_2-y_2)\left(\frac{\cosh\rho}{\cosh^2\rho-\cos^2\phi} - \frac{\cosh\rho'}{\cosh^2\rho'-\cos^2\phi'}\right).
\end{aligned}$$

Now

$$\begin{aligned}
\frac{\cosh\rho}{\cosh^2\rho-\cos^2\phi} - \frac{\cosh\rho'}{\cosh^2\rho'-\cos^2\phi'} &= \frac{\cosh\rho-\cosh\rho'}{\cosh^2\rho-\cos^2\phi} \\
&\qquad + \cosh\rho'\left(\frac{\cosh^2\rho'-\cosh^2\rho+\cos^2\phi-\cos^2\phi'}{(\cosh^2\rho-\cos^2\phi)(\cosh^2\rho'-\cos^2\phi')}\right).
\end{aligned}$$

Next we use again the expressions for $\cosh\rho-\cosh\rho'$ and $\cos\phi-\cos\phi'$ as before and for $x_2^2-y_2^2$ we use $B.2$.

A.5. **Expression for** $(x_2+y_2)(\xi_2-\eta_2)$**.** We have

$$\begin{aligned}
&\frac{(x_2+y_2)(\xi_2-\eta_2)}{\frac{4}{(1-\alpha)s}\omega}\\
&\quad=\frac{-x_2^2\cosh\rho}{\cosh^2\rho-\cos^2\phi}+\frac{y_2^2\cosh\rho'}{\cosh^2\rho'-\cos^2\phi'}\\
&\qquad+x_2y_2\left(\frac{\cosh\rho'}{\cosh^2\rho'-\cos^2\phi'}-\frac{\cosh\rho}{\cosh^2\rho-\cos^2\phi}\right)\\
&\quad=(y_2^2-x_2^2)\frac{\cosh\rho}{\cosh^2\rho-\cos^2\phi}\\
&\qquad+y_2(x_2+y_2)\left(\frac{\cosh\rho}{\cosh^2\rho-\cos^2\phi}-\frac{\cosh\rho'}{\cosh^2\rho'-\cos^2\phi'}\right).
\end{aligned}$$

Now we are in a similar situation as in the previous case.

A.6. **Expression for** $\xi_2^2-\eta_2^2$**.** We have

$$\begin{aligned}
\frac{\xi_2^2-\eta_2^2}{(\frac{4\omega}{(1-\alpha)s})^2}&=\left(\frac{x_2^2\cosh^2\rho}{\cosh^2\rho-\cos^2\phi}-\frac{y_2^2\cosh^2\rho'}{\cosh^2\rho'-\cos^2\phi'}\right)\\
&=\frac{(x_2^2-y_2^2)\cosh^2\rho}{\cosh^2\rho-\cos^2\phi}+y_2^2\left(\frac{\cosh^2\rho}{\cosh^2\rho-\cos^2\phi}-\frac{\cosh^2\rho'}{\cosh^2\rho'-\cos^2\phi'}\right)\\
&=\frac{(x_2^2-y_2^2)\cosh^2\rho}{\cosh^2\rho-\cos^2\phi}+y_2^2\Bigg(\frac{\cosh^2\rho-\cosh^2\rho'}{\cosh^2\rho-\cos^2\phi}+\\
&\qquad\cosh^2\rho'\frac{(\cosh^2\rho'-\cosh^2\rho)+(\cos^2\phi-\cos^2\phi')}{(\cosh^2\rho'-\cos^2\phi')(\cosh^2\rho-\cos^2\phi)}\Bigg)
\end{aligned}$$

This completes this part.

## Appendix B. Expressions for $t_s^-$ and $t_s^+$

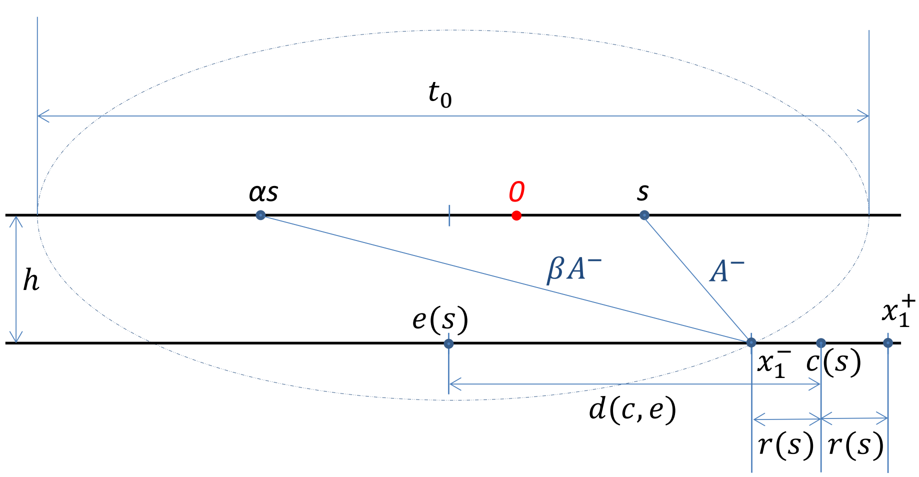


Figure 5. The geometric setup of filtering, showing the vertical cross-section corresponding to $x_2=0$.

Recall that $\Sigma_2$ is defined in (2.13) as

$$\Sigma_2 = \left\{(s,x,\omega) \in \mathcal{C}_{\mathcal{G}} : \left(x_1 - \frac{2\alpha s}{\alpha+1}\right)^2 + x_2^2 = -\alpha s^2 \frac{(\alpha-1)^2}{(\alpha+1)^2} - h^2\right\}$$

Recall that $s_0$ is defined by (5.5) and for $s > s_0$ $\Sigma_2$ is nonempty and not trivial.

We assume in this section that the cutoff function $f$ in Section 2 is chosen so it is zero for $s \le s_0$.

The radius and the $x_1$-coordinate of the center of circle $\Sigma_2$ are

$$r(s) = \sqrt{\frac{-\alpha s^2(\alpha-1)^2}{(\alpha+1)^2} - h^2}, \text{ and } c(s) = \frac{2\alpha s}{\alpha+1}.$$

Let $e(s) = (\alpha+1)s/2 < 0$ denote the center of ellipses on the plane. Then the distance between $e(s)$ and $c(s)$ can be written as

$$d(c,e) = -\frac{s(\alpha-1)^2}{2(\alpha+1)}.$$

For a fixed $s$ let $t_s^-$ and $t_s^+$ denote correspondingly the smallest and the largest values of $t$, for which the ellipsoid intersects $\Sigma_2$. Notice, that since the normal to an ellipse at a point $P$ bisects the angle from the $P$ to the foci, the condition $\widetilde{\gamma_R}(s) \le \gamma_R(s) < c(s)$ implies that our ellipses on the ground can not intersect the circle $\Sigma_2$ at more than two points. Here $\widetilde{\gamma_R}(s)$ denotes the right focus of the ellipse on the ground. Figure 5 shows the setup for $t_0$, where the ellipsoid passes through $x_1^-$, the closest to $e(s)$ point of $\Sigma_2$. The setup for $t_s^+$ is similar, with the ellipsoid passing through $x_1^+$, which is the farthest from $e(s)$ point of $\Sigma_2$.

A straightforward computation shows that

$$t_s^- = 2(\beta+1)\sqrt{\frac{d(d-r)}{\beta^2+1}}, \quad t_s^+ = 2(\beta+1)\sqrt{\frac{d(d+r)}{\beta^2+1}}, \tag{B.1}$$

where $\beta = \sqrt{-\alpha}$.

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